

# Proving with Computer Assistance, 2IMF15

Herman Geuvers, TUE

## Exercises on Simple Type Theory à la Curry: assigning types to untyped terms, principal type algorithm

### On simple type theory à la Church

1. Consider the following types

$$\begin{aligned} A_1 &= ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha \\ A_2 &= ((\alpha \rightarrow \alpha) \rightarrow \beta) \rightarrow \beta \\ A_3 &= ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \beta \end{aligned}$$

For only one of these types  $A_i$  there is a closed term  $M$  of type  $A_i$ . Pick the right  $A_i$ , construct  $M$  and give a typing derivation of  $M : A_i$ .

**Answer:** .....  
(partial): For  $A_2$  we can find a closed term. A term of type  $A_2$  is  $\lambda f : (\alpha \rightarrow \alpha) \rightarrow \beta. f (\lambda x : \alpha. x)$   
**End Answer** .....

2. In case you haven't done these yet:

- (a) Find a term of type  $(\delta \rightarrow \delta \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\delta \rightarrow \beta) \rightarrow \delta \rightarrow \gamma$
- (b) Find two terms of type  $(\delta \rightarrow \delta \rightarrow \alpha) \rightarrow (\gamma \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \delta \rightarrow \gamma \rightarrow \beta$
- (c) Find a term of type  $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha$
- (d) Find a term of type  $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \beta$  (Hint: use the previous exercise.)

### On simple type theory à la Curry

1. Determine the most general unifiers of

- (a)  $(\alpha \rightarrow \beta) \rightarrow \gamma$  and  $\alpha \rightarrow \beta \rightarrow \gamma$
- (b)  $(\alpha \rightarrow \beta) \rightarrow \gamma$  and  $\gamma \rightarrow \alpha \rightarrow \beta$

**Answer:** .....  
For clarity I first insert the brackets:

- (a)  $(\alpha \rightarrow \beta) \rightarrow \gamma \simeq \alpha \rightarrow (\beta \rightarrow \gamma)$ , yielding  $\alpha \rightarrow \beta \simeq \alpha$  and  $\gamma \simeq \beta \rightarrow \gamma$ . The first equation can't be fulfilled, so : "Fail".
- (b)  $(\alpha \rightarrow \beta) \rightarrow \gamma \simeq \gamma \rightarrow (\alpha \rightarrow \beta)$ , yielding  $\alpha \rightarrow \beta \simeq \gamma$  and  $\gamma \simeq \alpha \rightarrow \beta$ , yielding as a solution  $\gamma := \alpha \rightarrow \beta$ . This substitution is the most general unifier.

**End Answer** .....

2. Compute the principal type of  $\mathbf{S} := \lambda x. \lambda y. \lambda z. x z (y z)$ .

**Answer:** .....

We assign type variables to all abstracted variables and applicative sub-terms:

$$\lambda x:\alpha. \lambda y:\beta. \lambda z:\gamma. ((x z)^\delta (y z)^\epsilon)^\zeta.$$

The type of this term is now  $\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \zeta$ .

To make this term typable, we have to solve the following equations.

$$\begin{aligned} \alpha &\simeq \gamma \rightarrow \delta \\ \beta &\simeq \gamma \rightarrow \epsilon \\ \delta &\simeq \epsilon \rightarrow \zeta. \end{aligned}$$

Substituting the third gives

$$\begin{aligned} \alpha &\simeq \gamma \rightarrow \epsilon \rightarrow \zeta \\ \beta &\simeq \gamma \rightarrow \epsilon \\ \delta &\simeq \epsilon \rightarrow \zeta. \end{aligned}$$

This has the shape of a substitution (only variables to the right; none of the variables at the right occurs at the left), so we have found the most general unifier.

The principal type is now (substituting in the type  $\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \zeta$  above):

$$(\gamma \rightarrow \epsilon \rightarrow \zeta) \rightarrow (\gamma \rightarrow \epsilon) \rightarrow \gamma \rightarrow \zeta$$

**End Answer** .....

3. Which of the following terms is typable? If it is, determine the *principal type*; if it isn't, show that the typing algorithm fails.

(a)  $\lambda z x. z(x(\lambda y. y x))$

**Answer:** .....

We assign type variables to all abstracted variables and applicative subterms:

$$\lambda z:\alpha. \lambda x:\beta. (z(x(\lambda y:\gamma. (y x)^\delta)))^\epsilon)^\zeta$$

The type of this term is now  $\alpha \rightarrow \beta \rightarrow \zeta$ .

To make this term typable, we have to solve the following equations.

$$\begin{aligned} \gamma &\simeq \beta \rightarrow \delta \\ \beta &\simeq (\gamma \rightarrow \delta) \rightarrow \epsilon \\ \alpha &\simeq \epsilon \rightarrow \zeta \end{aligned}$$

Substituting the second, we get

$$\begin{aligned}\gamma &\simeq ((\gamma \rightarrow \delta) \rightarrow \epsilon) \rightarrow \delta \\ \beta &\simeq (\gamma \rightarrow \delta) \rightarrow \epsilon \\ \alpha &\simeq \epsilon \rightarrow \zeta\end{aligned}$$

The first equation is not solvable, so: “Fail”.

**End Answer** .....

(b)  $\lambda z x. z(x(\lambda y. y z))$

**Answer:** .....

We assign type variables to all abstracted variables and applicative subterms:

$$\lambda z: \alpha. \lambda x: \beta. (z(x(\lambda y: \gamma. (y z)^\delta))^\epsilon)^\zeta$$

The type of this term is now  $\alpha \rightarrow \beta \rightarrow \zeta$ .

To make this term typable, we have to solve the following equations.

$$\begin{aligned}\gamma &\simeq \alpha \rightarrow \delta \\ \beta &\simeq (\gamma \rightarrow \delta) \rightarrow \epsilon \\ \alpha &\simeq \epsilon \rightarrow \zeta\end{aligned}$$

Substituting the first, we get

$$\begin{aligned}\gamma &\simeq \alpha \rightarrow \delta \\ \beta &\simeq ((\alpha \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \\ \alpha &\simeq \epsilon \rightarrow \zeta\end{aligned}$$

Substituting the last, we get

$$\begin{aligned}\gamma &\simeq (\epsilon \rightarrow \zeta) \rightarrow \delta \\ \beta &\simeq (((\epsilon \rightarrow \zeta) \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \\ \alpha &\simeq \epsilon \rightarrow \zeta\end{aligned}$$

and we have our most general unifier. So, the principal type is

$$(\epsilon \rightarrow \zeta) \rightarrow (((\epsilon \rightarrow \zeta) \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \rightarrow \zeta.$$

**End Answer** .....

4. Consider the following two terms

- $\lambda x. x (\lambda y. y (\lambda z. x))$
- $\lambda x. x (\lambda y. x (\lambda z. z))$

For each of these terms, compute its principal type, if it exists. (Give the end result and show your computation; if the term has no principal type, show how your computation yields ‘fail’.)

5. Consider the following two terms

- $(\lambda x. \lambda y. x(\lambda z. y)) (\lambda w. w)$
- $(\lambda x. \lambda y. y(\lambda z. y)) (\lambda w. w)$

For each of these terms, compute its principal, if it exists. (Give the end result and show your computation; if the term has no principle type, show how your computation yields ‘fail’.)

**Answer:** .....  
 (partial): the first is typable, the second is not  
**End Answer** .....

6. Compute the principal type of  $M := \lambda x. \lambda y. x(y(\lambda z. x z z))(y(\lambda z. x z z))$ .

7. For each of the following two terms, compute its principal type, if it exists.

- $\lambda x. (\lambda y. x(x y)) (\lambda u v. u)$
- $\lambda y. (\lambda x. x(x y)) (\lambda u v. u)$

Give the end result and show your computation; if the term has no principal type, show how your computation yields ‘fail’.

**Answer:** .....  
 (partial): The first one is typable, the second is not. We show the computation of the principal type for the first term. We annotate the abstracted variables and the applicative subterms with type variables:

$$\lambda x^A. ((\lambda y^B. (x(x y)^E)^F) (\lambda u^C v^D. u))G \quad : \quad A \rightarrow G$$

We generate the set of equations that we need to solve:

$$\left\{ \begin{array}{l} A = B \rightarrow E \\ A = E \rightarrow F \\ B \rightarrow F = (C \rightarrow D \rightarrow C) \rightarrow G \end{array} \right.$$

We solve it:

$$\left\{ \begin{array}{l} A = B \rightarrow E \\ A = E \rightarrow F \\ B \rightarrow F = (C \rightarrow D \rightarrow C) \rightarrow G \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} A = B \rightarrow E \\ A = E \rightarrow F \\ B = C \rightarrow D \rightarrow C \\ F = G \end{array} \right. \Leftrightarrow$$

$$\left\{ \begin{array}{l} A = (C \rightarrow D \rightarrow C) \rightarrow E \\ A = E \rightarrow G \\ B = C \rightarrow D \rightarrow C \\ F = G \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} A = E \rightarrow G \\ E \rightarrow G = (C \rightarrow D \rightarrow C) \rightarrow E \\ B = C \rightarrow D \rightarrow C \\ F = G \end{array} \right. \Leftrightarrow$$

$$\left\{ \begin{array}{l} A = E \rightarrow G \\ E = C \rightarrow D \rightarrow C \\ G = E \\ B = C \rightarrow D \rightarrow C \\ F = G \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} A = (C \rightarrow D \rightarrow C) \rightarrow G \\ E = C \rightarrow D \rightarrow C \\ G = C \rightarrow D \rightarrow C \\ B = C \rightarrow D \rightarrow C \\ F = G \end{array} \right. \Leftrightarrow$$

$$\left\{ \begin{array}{l} A = (C \rightarrow D \rightarrow C) \rightarrow C \rightarrow D \rightarrow C \\ E = C \rightarrow D \rightarrow C \\ G = C \rightarrow D \rightarrow C \\ B = C \rightarrow D \rightarrow C \\ F = C \rightarrow D \rightarrow C \end{array} \right.$$

This is a substitution, so we are done. Applying this substitution to the type  $A \rightarrow G$ , we get the principal type, which is:

$$((C \rightarrow D \rightarrow C) \rightarrow C \rightarrow D \rightarrow C) \rightarrow C \rightarrow D \rightarrow C.$$

**End Answer** .....