Semantics and Domain theory

Exercises 7

1. Show that the untyped λ-term ω ($= \lambda x.x\,x$) is not typable in PCF. That is: show that there are no τ_1 and τ_2 such that $\vdash \mathbf{fn}\; x : \tau_1.\, x\, x \;\; : \;\; \tau_2$.

2. (a) Suppose that the term mult : $\mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}$ defines multiplication in PCF. Give a PCF term that defines the exponentiation function exp : $\mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}$. (So $\exp n\, m$ should denote n^m.)

   (b) Let $p : \mathbf{nat} \to \mathbf{nat}$. Define a term $N : \mathbf{nat}$ that denotes the smallest number n such that $p(n) = 0$ and $\forall i < n(p(n) > 0)$.

3. Show that, if $Q \Downarrow_\tau V$, then $(\mathbf{fn}\; x : \tau.\, \mathbf{fn}\; y : \tau.y)PQ \Downarrow V$. (NB. V denotes an arbitrary *value*.)

4. To prove that PCF evaluation is deterministic, we prove (in Proposition 5.4.1) that the following set is closed under the rules of Fig.3

$$\{(M, \tau, V) | M \Downarrow_\tau V \wedge \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')\}$$

   Show this for the cases of the rules ($\Downarrow_{\mathrm{if1}}$) and ($\Downarrow_{\mathrm{cbn}}$).
   (An alternative way of looking at this is to prove the following:

$$M \Downarrow_\tau V \Rightarrow \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')$$

   by induction on the derivation of $M \Downarrow_\tau V$. Do only the cases when the last applied rule is ($\Downarrow_{\mathrm{if1}}$) or ($\Downarrow_{\mathrm{cbn}}$).)

5. Given the definition of plus (Exercise 5.6.3.)

$$\begin{aligned} \text{plus} \;\; = \;\; & \mathbf{fix}(\mathbf{fn}\; p : \mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}.\, \mathbf{fn}\; x : \mathbf{nat}.\, \mathbf{fn}\; y : \mathbf{nat}. \\ & \;\; \mathbf{if}\; \mathbf{zero}(y)\; \mathbf{then}\; x\; \mathbf{else}\; \mathbf{succ}(p\, x\, \mathbf{pred}(y))) \end{aligned}$$

   Prove (by induction) that

$$\forall m, n(\text{plus}\; \mathbf{succ}^m(0)\; \mathbf{succ}^n(0) \;\; \Downarrow_{\mathbf{nat}} \;\; \mathbf{succ}^{m+n}(0))$$

   NB. First identify the proper statement that you need and that you can prove relatively easily by induction.