Exercises Coalgebra for Lecture 5

The exercises labeled with (*) are optional and more advanced.

1. What are initial/final objects in the following categories (if they exist)?
   (a) the *discrete category* for a given set X; objects are elements of X, and the only arrows are the identity arrows;
   (b) the category Cat: objects are small categories,[1] arrows are functors;
   (c) $\mathsf{SetsRel}$ (recall: objects are sets, arrows are relations);
2. Describe products and coproducts in each of the following categories (if they exist):
   (a) the discrete category for a given set X;
   (b) the category Cat (see previous exercise);
   (c) the category Pred: an object of Pred is a pair (P, X) of sets with $P \subseteq X$, and an arrow from an object (P, X) to an object (Q, Y) is a map $f\colon X \to Y$ such that for all $x \in P$: $f(x) \in Q$.
3. Let $\mathcal{C}$ be a category with products (that means the product of any two objects in $\mathcal{C}$ exists) and a final object 1. Show that $1 \times X \cong X$.
4. Let Poset be the category whose objects are partially ordered sets, and arrows are monotone maps. Show that there is a functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Poset}$ which maps a set X to the partial order $(\mathcal{P}(X), \subseteq)$.
5. Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor on a category $\mathcal{C}$. An F-*algebra* is a pair $(X, a\colon F(X) \to X)$, where X is an object in $\mathcal{C}$ and a an arrow.
   (a) Define the notion of algebra homomorphism (dual to that of a coalgebra homomorphism, that is, with all arrows reversed) and show that F-algebras and homomorphisms between them form a category $\mathsf{Alg}(F)$.
   (b) An *initial algebra* is an initial object in the category $\mathsf{Alg}(F)$. Show that, if (X, a) is an initial algebra, then a is an isomorphism.
6. (*) We would like to define a functor $S\colon \mathsf{Set} \to \mathsf{Set}$ by $S(X) = X^\omega$, i.e., a set X is mapped to the set of streams over X.
   (a) Define S on a function $f\colon X \to Y$, using that Y^ω is the final stream system over Y; $S(f)$ should apply f to all elements of a given stream.
   (b) Show that S is functorial.
7. (*) Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor on a category $\mathcal{C}$. Suppose $\mathcal{C}$ has an initial object, and a coproduct $X + Y$ for any objects X, Y. Show that $\mathsf{CoAlg}(F)$ has an initial object and all coproducts as well.

[1] A category is small if its objects form a set, not a proper class. That's a foundational point, feel free to ignore it for this exercise.