1 Counting Letters

a) Let A be a non-empty, finite alphabet and $a \in A$ a letter.

Define by structural induction a map **(1pt)**

$$|\,.\,|_a : A^* \to \mathbb{N}$$

that counts the number of occurrences of the letter a in a word.

Solution

We define $|\,w\,|_a$ by induction on w:

- *Base case: $|\lambda|_a = 0$.*
- *Step case for $xv \in A^*$. So we have already defined $|v|_a$. We distinguish two cases: If $x = a$, then $|av|_a = |v|_a + 1$, otherwise if $x \neq a$, then we define $|xv|_a = |v|_a$.*

b) Show, by induction, that for any two words $w, u \in A^*$ **(2pt)**

$$|wu|_a = |w|_a + |u|_a.$$

Solution

We show $|wu|_a = |w|_a + |u|_a$ by induction on w.

- *Base case: $w = \lambda$. Then we get*
$$|wu|_a = |u|_a = 0 + |u|_a = |w|_a + |u|_a.$$
- *Induction step: Let $w = xv$ for some $x \in A$. Induction Hypothesis: $|vu|_a = |v|_a + |u|_a$ holds for all $u \in A^*$. We distinguish cases:*
If $x = a$, then
$$|avu|_a = 1 + |vu|_a \overset{IH}{=} 1 + |v|_a + |u|_a = |av|_a + |u|_a = |w|_a + |u|_a.$$
If $x \neq a$, then
$$|xvu|_a = |vu|_a \overset{IH}{=} |v|_a + |u|_a = |xv|_a + |u|_a = |w|_a + |u|_a.$$

This induction shows, that $|wu|_a = |w|_a + |u|_a$ for all $w, u \in A^*$.

2 Regular expression

Let L be the language given by

$$\left\{ w \in \{a, b\}^* \,\middle|\, \text{every } b \text{ in } w \text{ is directly followed by an } a \right\}$$

Give a regular expression for the language L and explain your answer. **(1pt)**

Solution

We use $a^*(baa^*)^*$.

Explanation: A word in the language should consist of a finite (possibly zero!) number of ba's that can be interleaved with a's, so a word in L is of the shape $a^{n_1}(ba)a^{n_2}\dots(ba)a^{n_k}$ *with* $n_1, n_2 \dots n_k \geq 0$. *This is captured by the regular expression* $a^*(baa^*)^*$.

Obvious alternatives: $a^*(a^*baa^*)^*$ *or* $(a^*ba)^*a^*$.