1 Counting Letters

a) Let A be a non-empty, finite alphabet and $a \in A$ a letter.

Define by structural induction a map **(1pt)**

$$|\,.\,|_a \colon A^* \to \mathbb{N}$$

that counts the number of occurrences of the letter a in a word.

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Solution

We define $|\,w\,|_a$ *by induction on* w*:*

- *Base case:* $|\lambda|_a = 0$.
- *Step case for* $xv \in A^*$*. So we have already defined* $|v|_a$*. We distinguish two cases: If* $x = a$*, then* $|av|_a = |v|_a + 1$*, otherwise if* $x \neq a$*, then we define* $|xv|_a = |v|_a$.

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b) Show, by induction, that for any two words $w, u \in A^*$ **(2pt)**

$$|wu|_a = |w|_a + |u|_a.$$

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Solution

We show $|wu|_a = |w|_a + |u|_a$ *by induction on* w.

- *Base case:* $w = \lambda$*. Then we get*

$$|wu|_a = |u|_a = 0 + |u|_a = |w|_a + |u|_a.$$

- *Induction step: Let* $w = xv$ *for some* $x \in A$*. Induction Hypothesis:* $|vu|_a = |v|_a + |u|_a$ *holds for all* $u \in A^*$*. We distinguish cases:*
  If $x = a$*, then*

$$|avu|_a = 1 + |vu|_a \overset{IH}{=} 1 + |v|_a + |u|_a = |av|_a + |u|_a = |w|_a + |u|_a.$$

  If $x \neq a$*, then*

$$|xvu|_a = |vu|_a \overset{IH}{=} |v|_a + |u|_a = |xv|_a + |u|_a = |w|_a + |u|_a.$$

This induction shows, that $|wu|_a = |w|_a + |u|_a$ *for all* $w, u \in A^*$.

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2 Regular expression

Let L be the language given by

$$\left\{ w \in \{a,b\}^* \,\middle|\, \text{every } b \text{ in } w \text{ is directly followed by an } a \right\}$$

Give a regular expression for the language L and explain your answer. **(1pt)**

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Solution

We use $a^*(baa^*)^*$.

Explanation: A word in the language should consist of a finite (possibly zero!) number of ba's that can be interleaved with a's, so a word in L is of the shape $a^{n_1}(ba)a^{n_2}\dots(ba)a^{n_k}$ *with* $n_1, n_2 \dots n_k \geq 0$. *This is captured by the regular expression* $a^*(baa^*)^*$.

Obvious alternatives: $a^*(a^*baa^*)^*$ *or* $(a^*ba)^*a^*$.

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