Exercises Coalgebra for Lecture 2

Given $a \in A^\omega$, a^ω denotes the constant stream $(a, a, a, \ldots)$.

1. Consider the function $\mathit{double}\colon A^\omega \to A^\omega$ given by

$$\mathit{double}(\sigma) = (\sigma(0), \sigma(0), \sigma(1), \sigma(1), \ldots)\,.$$

   (a) Define *double* by stream differential equations (in terms of initial value and derivative).

   (b) Give a stream system on the set $\{0,1\} \times A^\omega$, such that, given the unique homomorphism $h\colon \{0,1\} \times A^\omega \to A^\omega$ to the final stream system, we have $h(0,\sigma) = \mathit{double}(\sigma)$ for all $\sigma \in A^\omega$.
   If you find this difficult, it may help to make exercise 5 first.

2. Prove, by defining suitable bisimulations, that for all $\sigma, \tau \in A^\omega$:

   (a) $\mathit{odd}(\sigma) = \mathit{even}(\sigma')$

   (b) $\mathit{double}(\sigma) = \mathit{zip}(\sigma, \sigma)$

   (c) $\mathit{odd}(\mathit{zip}(\sigma, \tau)) = \tau$

   (d) $\mathit{zip}(a^\omega, b^\omega) = (ab)^\omega$ for some $a, b \in A$

3. In this exercise we consider streams over the natural numbers: $\mathbb{N}^\omega$. The (pointwise) *sum* $\sigma \oplus \tau$ of streams $\sigma, \tau \in \mathbb{N}^\omega$ is given by $(\sigma \oplus \tau)(n) = \sigma(n) + \tau(n)$.
   We define two functions $f, g\colon A^\omega \to A^\omega$ by stream differential equations:

$$\begin{array}{rclcrcl} f(\sigma)(0) & = & \sigma(0) & \quad & g(\sigma)(0) & = & \sigma(0) \\ f(\sigma)' & = & f(\sigma) \oplus \sigma' & & g(\sigma)' & = & (\sigma(0))^\omega \oplus g(\sigma') \end{array}$$

   (a) Characterise $\oplus$ by stream differential equations.

   (b) What does f compute? Explain what $f(\sigma)$ is, given a stream σ. Similarly for g.

   (c) **(*)** Show that $f = g$, using a suitable bisimulation.

4. Recall the following coinduction principle: if $R \subseteq A^\omega \times A^\omega$ is a bisimulation, then $(\sigma, \tau) \in R$ implies $\sigma = \tau$. Complete the proof given in the lecture by showing that, if R is a bisimulation, then $(\sigma, \tau) \in R$ implies $(\sigma^{(i)}, \tau^{(i)}) \in R$ for all i (first define $\sigma^{(i)} = (\sigma(i), \sigma(i+1), \sigma(i+2), \ldots)$ properly by induction).

5. Consider the stream system $(A^\omega, \langle o, t\rangle)$ defined by

$$o(\sigma) = \sigma(0) \qquad\qquad t(\sigma) = \sigma''\,.$$

   Then there is a unique homomorphism $h\colon A^\omega \to A^\omega$ from $(A^\omega, \langle o, t\rangle)$ to $(A^\omega, \langle i, d\rangle)$, as we saw in the lecture (and see exercise 6). Spell out the details: what does it mean for h to be a homomorphism? What is h?

6. In the lecture, we have seen that for every stream system $(X, \langle o, t \rangle)$, there exists a map $h\colon X \to A^\omega$, which we claimed to be a homomorphism from $(X, \langle o, t \rangle)$ to $(A^\omega, \langle i, d \rangle)$ (recall that $i(\sigma) = \sigma(0)$ and $d(\sigma) = \sigma'$). This map h is given by

$$h(x) = (o(x), o(t(x)), o(t(t(x))), \ldots) .$$

Formally, it is defined by $h(x)(i) = o(t^i(x))$, where $t^0(x) = x$ and $t^{i+1}(x) = t(t^i(x))$.

Finish the proof that (A^ω, i, d) is a final stream system, by showing:

(a) h, as defined above, is indeed a homomorphism;

(b) if k, l are homomorphisms from $(X, \langle o, t \rangle)$ to $(A^\omega, \langle i, d \rangle)$ then $k = l$.

7. **(*)** (Exercise 80 in [Rutten]). Show that the stream system $(A^\omega \times A^\omega, \langle o, t \rangle)$, defined by $o((\sigma, \tau)) = \sigma(0)$ and $t(\sigma, \tau) = (\tau, \sigma')$, for all $(\sigma, \tau) \in A^\omega \times A^\omega$, is final.