Exercises Coalgebra for Lecture 4

The exercises labeled with (*) are optional and more advanced.

1. We would like to define a category Pred of "predicates". An object of Pred is a pair (P, X) of sets with $P \subseteq X$, and an arrow from an object (P, X) to an object (Q, Y) is a map $f : X \to Y$ such that for all $x \in P$: $f(x) \in Q$.

   (a) Show that Pred is a category, by defining suitable identity arrows and composition, and checking that the required laws are satisfied.

   (b) Are there any functors from Pred to Set? If so, choose your favorite and show it is a functor indeed.

2. Describe products and coproducts in the following categories, if they exist:

   (a) The category Cat of categories and functors.[1]

   (b) A preorder $(P, \sqsubseteq)$ seen as a category (objects are elements of P, and there is an arrow $x \to y$ if and only if $x \sqsubseteq y$).

3. What are initial/final objects in the following categories (if they exist)?

   (a) $\mathsf{SetsRel}$ (recall: objects are sets, arrows are relations);

   (b) the *discrete category* for a given set X; objects are elements of X, and the only arrows are the identity arrows;

   (c) the category Cat from Exercise 2a;

   (d) a preorder $(P, \sqsubseteq)$ seen as a category (Exercise 2b);

   (e) (*) the category Mon whose objects are monoids (see 7) and whose arrows are monoid homomorphisms; a homomorphism from $(M, \cdot_M, 1_M)$ to $(N, \cdot_N, 1_N)$ is a function $h : M \to N$ such that for all $m, n \in M$: $h(m \cdot_M n) = h(m) \cdot_N h(n)$ and $h(1_M) = 1_N$.

4. Recall that two objects X, Y in a category $\mathcal{C}$ are *isomorphic*, written $X \cong Y$, if there is an isomorphism $f : X \to Y$, that is, an arrow $f : X \to Y$ with another arrow $g : Y \to X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$.

   (a) Show that any functor $F : \mathcal{C} \to \mathcal{D}$ preserves isomorphisms: if $X \cong Y$ then $F(X) \cong F(Y)$.

   (b) Show that $\cong$ is an equivalence relation.

5. Let $\mathcal{C}$ be a category which has products (that is, the product $X \times Y$ exists for all $X, Y \in \mathsf{Ob}(\mathcal{C})$ and a final object 1.

   (a) Prove that $X \times 1 \cong X$.

   (b) (*) Suppose $\mathcal{C}$ also has an initial object 0. Do we have $X \times 0 \cong 0$? Give either a proof or a counterexample.

[1] The categories being "small", meaning they have only a set (not a proper class) of objects: feel free to ignore this if that makes no sense to you.

6. Let $\mathcal{C}$ be a category. We define the *opposite category* $\mathcal{C}^{\mathsf{op}}$ as the category which has the same objects as $\mathcal{C}$, but where all arrows are reversed: thus, $f\colon X \to Y$ is an arrow in $\mathcal{C}^{\mathsf{op}}$ iff $f\colon Y \to X$ is an arrow in $\mathcal{C}$.

   (a) How should composition in $\mathcal{C}^{\mathsf{op}}$ be defined? And identity arrows? Show, in detail, that $\mathcal{C}^{\mathsf{op}}$ is a category.

   (b) Show that an object 0 is initial in $\mathcal{C}$ iff it is final in $\mathcal{C}^{\mathsf{op}}$.

   (c) **(*)** Show that $\mathsf{SetsRel} \cong \mathsf{SetsRel}^{\mathsf{op}}$.

7. **(*)** A *monoid* is a triple $(M, \cdot, 1)$ where M is a set, $\cdot$ is a binary operation and $1 \in M$ an element, such that for all $m, n, p \in M$: $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ and $m \cdot 1 = m = 1 \cdot m$.

   (a) Show that a monoid corresponds to a one-object category.

   (b) Let $(M, \cdot, 1)$ be a monoid, represented as a category M as in the previous exercise. Show that a functor $F\colon M \to \mathsf{Set}$ corresponds to a *monoid action*: a set X together with a function $\mu\colon M \to X^X$ (where X^X is the set of functions from X to X) such that for all $x \in X$: $\mu(1)(x) = x$ and for all $m, n \in M$: $\mu(m \cdot n)(x) = \mu(m)(\mu(n)(x))$.