Exercises Coalgebra for Lecture 5

The exercises labeled with **(*)** are optional and more advanced.

1. Let A be a set, and consider the functor $F\colon \mathsf{Set} \to \mathsf{Set}$, defined on a set X by $F(X) = A \times X \times X$ and on a function f by $F(f) = \mathsf{id}_A \times f \times f$. An *infinite binary tree* (node-labelled in A) is a function $t\colon \{l, r\}^* \to A$, where $\{l, r\}^*$ is the set of words over l, r. The empty word is denoted by $\varepsilon \in \{l, r\}^*$. The set $A^{\{l,r\}^*}$ of all infinite binary trees is denoted by T.

   Given a tree $t \in T$, we define $t_l, t_r \in T$ as follows: $t_l(w) = t(lw)$ and $t_r(w) = t(rw)$, for all $w \in \{l, r\}^*$. Consider the F-coalgebra $z\colon T \to A \times T \times T$ defined by
   $$z(t) = (t(\varepsilon), t_l, t_r)$$

   The aim of this exercise is to show that (T, z) is a final F-coalgebra.

   (a) Describe, in words, what t_l and t_r are, given a tree $t \in T$.

   (b) Let $\langle g_\varepsilon, g_1, g_2\rangle\colon X \to A \times X \times X$ be an F-coalgebra. Show that a map $h\colon X \to T$ is a homomorphism from $(X, \langle g_\varepsilon, g_1, g_2\rangle)$ to (T, z) if and only if for all $x \in X$:

      i. $h(x)(\varepsilon) = g_\varepsilon(x)$

      ii. $h(x)_l = h(g_1(x))$

      iii. $h(x)_r = h(g_2(x))$

   (c) Conclude that such a homomorphism (as in the previous exercise) exists (why?).

   (d) Show that if h, k are both homomorphisms from (X, g) to (T, z) then $h = k$, by proving by induction on $w \in \{l, r\}^*$ that for all $x \in X$: $h(x)(w) = k(x)(w)$.

2. Consider the functor $F\colon \mathsf{Set} \to \mathsf{Set}$, defined on a set X by $F(X) = X + A$, where A is fixed set.

   (a) Make an (educated) guess about a final coalgebra for F. How would you define the unique homomorphism beh?

   (b) **(*)** Prove that your answer to the previous question is correct.

3. Let $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$. In the lecture, we defined the category $\mathsf{CoAlg}(F)$ whose objects are F-coalgebras, and whose morphisms are coalgebra morphisms. Show that this is indeed a category, by checking the necessary axioms.

4. In the lecture, we defined, for any given automaton $(S, \langle \epsilon, \delta\rangle)$, a map $\mathsf{beh}\colon S \to 2^{A^*}$.

   Finish the proof that $(2^{A^*}, \langle e, d\rangle)$ is a final coalgebra, by showing that beh is the unique coalgebra homomorphism from $(S, \langle \epsilon, \delta\rangle)$ to $(2^{A^*}, \langle e, d\rangle)$.

5. **(*)** Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor with a final coalgebra (Z, z). In the lecture, we defined two states $x, y \in X$ of an F-coalgebra (X, f) to be *behaviourally equivalent* if $\mathsf{beh}(x) = \mathsf{beh}(y)$, where beh is the unique homomorphism from (X, f) to (Z, z). Show that $\mathsf{beh}(x) = \mathsf{beh}(y)$ if and only if there exists an F-coalgebra (Y, g) and a homomorphism $h\colon X \to Y$ from (X, f) to (Y, g) such that $h(x) = h(y)$. Hint: draw a suitable diagram to clarify the situation.

6. **(*)** We would like to define a functor $S\colon \mathsf{Set} \to \mathsf{Set}$ by $S(X) = X^\omega$, i.e., a set X is mapped to the set of streams over X.

   (a) Define S on a function $f\colon X \to Y$, using that Y^ω is the final stream system over Y; $S(f)$ should apply f to all elements of a given stream.

   (b) Show that S is functorial.

7. **(*)** Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor on a category $\mathcal{C}$. Suppose $\mathcal{C}$ has an initial object, and a coproduct $X + Y$ for any objects X, Y. Show that $\mathsf{CoAlg}(F)$ has an initial object and all coproducts as well.