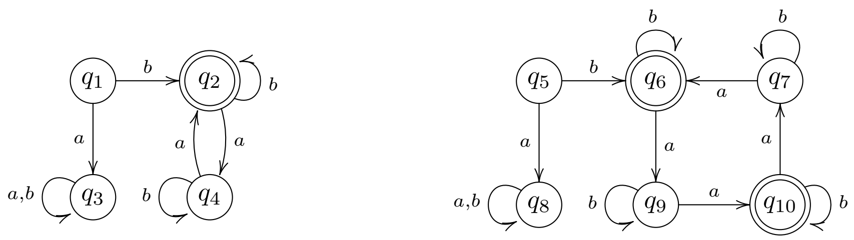
Exercises Coalgebra for Lecture 6

The exercises labeled with **(*)** are optional and more advanced.

1. Consider the following two deterministic automata. Show that states q_1 and q_5 accept the same language, by proving that they are bisimilar.



2. Consider the following stream system over some set A with $a, b \in A$:

$$x_1 \xrightarrow{a} x_2 \xrightarrow{b} x_3 \xrightarrow{a} x_4 \xrightarrow{b} x_5 \xrightarrow{a} x_6 \xrightarrow{b} \dots$$

   What is the largest bisimulation $\sim$ on it? What is the smallest?

3. Let A, B be sets, and define $F\colon \mathsf{Set} \to \mathsf{Set}$ by $F(X) = A \times X + B$ and $F(f) = \mathsf{id}_A \times f + \mathsf{id}_B$. For an F-coalgebra (X, g), $x, x' \in X$, $a \in A$, $b \in B$, we write $x \xrightarrow{a} x'$ if $g(x) = (a, x') \in A \times X$, and $x \downarrow b$ if $g(x) = b \in B$.

   (a) Instantiate the abstract notion of bisimulation between F-coalgebras to the above functor F, and spell out the details to obtain a concrete notion of bisimulation, formulated in terms of the notation $x \xrightarrow{a} x'$ and $x \downarrow b$.

   (b) Give an example of such an F-coalgebra, with two states that are bisimilar (show it with a suitable bisimulation) and two states that are not.

4. Let (X, f) be an F-coalgebra, for some functor $F\colon \mathsf{Set} \to \mathsf{Set}$. Show that the diagonal relation $\Delta_X = \{(x, x) \mid x \in X\}$ is a bisimulation on (X, f). Hint: use that $\mathsf{id}_X\colon X \to X$ is a homomorphism, and that there is a bijection $\Delta_X \cong X$.

5. Let $(X, \langle o, f\rangle)$ be a stream system over A, that is, a coalgebra for the functor $F(X) = A \times X$. As always, let $\mathsf{beh}\colon X \to A^\omega$ be the unique coalgebra homomorphism from $(X, \langle o, f\rangle)$ to the final coalgebra.

   Show that, for any two states $x, y \in X$, if $\mathsf{beh}(x) = \mathsf{beh}(y)$ then $x \sim y$.

6. **(*)** In the lecture, we have mentioned that homomorphisms preserve bisimilarity. In this exercise, we will investigate the converse. Concretely, for $F\colon \mathsf{Set} \to \mathsf{Set}$ a functor, we would like to prove (under certain conditions) that, if $h\colon (X, f) \to (Y, g)$ is a homomorphism of F-coalgebras, then the kernel relation

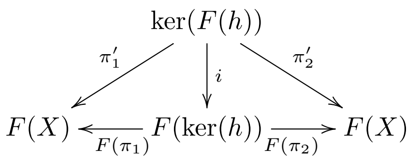
$$\ker(h) = \{(x, y) \in X \times X \mid h(x) = h(y)\}$$

is a bisimulation on (X, f). Below, we denote the projections of this relation by $\pi_1\colon \ker(h) \to X$ and $\pi_2\colon \ker(h) \to X$.

(a) Let F be an arbitrary functor, and $h\colon (X, f) \to (Y, g)$ a homomorphism of F-coalgebras. Let

$$\ker(F(h)) = \{(u, v) \in F(X) \times F(X) \mid F(h)(u) = F(h)(v)\}\,,$$

with projections $\pi'_1\colon \ker(F(h)) \to F(X)$ and $\pi'_2\colon \ker(F(h)) \to F(X)$. Suppose there is a map $i\colon \ker(F(h)) \to F(\ker(h))$ such that the following diagram commutes:



Show that $\ker(h)$ is a bisimulation on (X, f).

(b) Consider the functor $F\colon \mathsf{Set} \to \mathsf{Set}$ defined on a set X by

$$F(X) = \{(x, y, z) \mid \mathsf{card}\{x, y, z\} \leq 2\}$$

where $\mathsf{card}\{x, y, z\}$ is the number of elements of $\{x, y, z\}$; $F(X)$ consists of all triples of elements of X where at least two elements are equal. On a function f, the functor is defined by $F(f)(x, y, z) = (f(x), f(y), f(z))$. Show that any two states of a coalgebra for this functor are behaviourally equivalent.

(c) Let F be the functor from the previous exercise. Give an F-coalgebra on the set 2 with the property that there is no bisimulation $R \subseteq 2 \times 2$ such that $(0, 1) \in R$.

(d) Let F be the functor from the previous two exercises. Show that bisimilarity on F-coalgebras is not transitive, in general.