Coalgebra: homework assignment 1

October 18, 2018

If you have any questions, email me: `jrot@cs.ru.nl`. The deadline is Monday 5 November, 13:30 (lecture time). You can hand in by email, or in the lecture. Explain your answers!

Recall that A^ω denotes the set of streams over a set A. For an element $a \in A$, we denote by a^ω the stream consisting of only a's. Given $a \in A$ and $\sigma \in A^\omega$, concatenation is denoted by $a : \sigma$, defined by $(a : \sigma)(0) = a$ and $(a : \sigma)(n + 1) = \sigma(n)$ for all $n \in \mathbb{N}$.

1. Consider the function $\mathsf{flip}\colon A^\omega \to A^\omega$, given by

$$\mathsf{flip}(\sigma) = (\sigma(1), \sigma(0), \sigma(2), \sigma(1), \sigma(3), \sigma(2), \ldots)\,.$$

   (a) Characterise flip by stream differential equations (using initial value and derivative).

   (b) Show that $\mathsf{flip}(\sigma) = \mathsf{zip}(\sigma', \sigma)$ for all $\sigma \in A^\omega$, by constructing a suitable bisimulation.

2. Let A and B be sets. A *flexible and useful state-based system (FUSS)* consists of a set of states X, and for every $x \in X$ and $a \in A$ there is exactly one outgoing a-transition $x \xrightarrow{a} t$, where *either* $t = (x_1, x_2, x_3)$ for some $x_1, x_2, x_3 \in X$, *or* $t = b$ for some $b \in B$.

   (a) Give a functor $H\colon \mathsf{Set} \to \mathsf{Set}$ whose coalgebras correspond to FUSSes.

   (b) Derive a concrete notion of homomorphism between FUSSes from the notion of coalgebra homomorphism between H-coalgebras.

   (c) Describe, in words, the elements of a final H-coalgebra. A proof of finality is not needed.

3. We'd like to define a category Set_{inj}, where objects are sets, morphisms are *injective* functions and composition is given as in Set.

   (a) Show that Set_{inj} is indeed a category.

   (b) For each of the following, state whether it exists in Set_{inj}, and give a proof to justify your answer or give a counterexample: the coproduct of two sets X and Y; the product of two sets X and Y; an initial object; a final object.

   (c) Let $n \in \mathbb{N}$ be a natural number. Show that $\mathcal{P}_n\colon \mathsf{Set}_{inj} \to \mathsf{Set}_{inj}$, given by $\mathcal{P}_n(X) = \{S \subseteq X \mid S \text{ has exactly } n \text{ elements}\}$ on objects, can be extended to a functor.

   (d) Let $n \geq 1$, $c\colon X \to \mathcal{P}_n(X)$ be a coalgebra and $x, y \in X$ with $x \neq y$. Show that there are infinite, non-crossing paths that start in x and y respectively. Here, an infinite path in c that starts at x is a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 = x$ and $x_{n+1} \in c(x_n)$. Two paths $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are non-crossing if $x_n \neq y_n$ for all $n \in \mathbb{N}$.

4. Let $f\colon X \to \mathcal{P}(A \times X)$ be a labelled transition system (LTS) over a set A. We write $x \xrightarrow{a} x'$ iff $x' \in f(x)$. We define the *maximal trace semantics* $t\colon X \to 2^{A^*}$, for all $x \in X$, $a \in A$ and $w \in A^*$, as follows:

$$t(x)(\varepsilon) = \begin{cases} 1 & \text{if } f(x) = \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad t(x)(aw) = \begin{cases} 1 & \text{if } \exists x'.\, x \xrightarrow{a} x' \text{ and } t(x')(w) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Give an example of an LTS $f\colon X \to \mathcal{P}(A \times X)$ with two states $x, y \in X$ such that $t(x) = t(y)$, but x is *not* bisimilar to y (and show this).

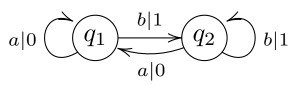
(b) Let $f\colon X \to \mathcal{P}(A \times X)$ be an LTS. Give a deterministic automaton $(\mathcal{P}(X), \langle o, \delta\rangle)$ such that $\mathsf{beh}(\{x\}) = t(x)$ for all $x \in X$, where $\mathsf{beh}\colon X \to 2^{A^*}$ is the language semantics of the automaton as defined in the lecture (you don't have to prove that your answer is correct).

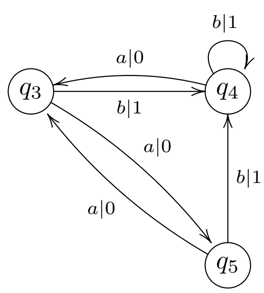
5. Let A, B be sets. Consider the functor $M\colon \mathsf{Set} \to \mathsf{Set}$ defined on sets by $M(X) = (B \times X)^A$ and on functions by $M(h) = (\mathsf{id}_B \times h)^A$. Coalgebras for M are so-called *Mealy machines*. Thus, a Mealy machine is a pair (S, f) where S is a set of states and $f\colon S \to (B \times S)^A$. Typically, transitions are represented by

$$x \xrightarrow{a|b} y \iff f(x)(a) = (b, y).$$

(a) Instantiate the coalgebraic definition of bisimulation for M to obtain a concrete description of bisimulations $R \subseteq S \times S$ for Mealy machines. Use thereby the notation $x \xrightarrow{a|b} y$ we introduced above.

(b) Consider the following two Mealy machines where $A = \{a, b\}$ and $B = \{0, 1\}$. Show that states q_1 and q_3 are bisimilar, using the answer to the previous question.





(c) Given streams σ and τ and a natural number $n \in \mathbb{N}$, we say σ and τ are equal up to n, denoted $\sigma \equiv_n \tau$, if for all i with $0 \le i < n$: $\sigma(i) = \tau(i)$. A map $\varphi\colon A^\omega \to B^\omega$ is called *causal* if for all $\sigma, \tau \in A^\omega$ and all $n \in \mathbb{N}$: $\sigma \equiv_n \tau$ implies $\varphi(\sigma) \equiv_n \varphi(\tau)$. Give an example of a function that is causal, and one that is not.

(d) Let Z be the set of all causal functions from A^ω to B^ω. Define the Mealy machine $z\colon Z \to (B \times Z)^A$ as follows, for all $\varphi \in Z$ and $a \in A$:

$$z(\varphi)(a) = (\varphi(a^\omega)(0), \psi)\,,$$

where $\psi\colon A^\omega \to B^\omega$ is defined by $\psi(\tau) = \varphi(a : \tau)'$, i.e., $\psi(\tau)(n) = \varphi(a : \tau)(n+1)$. Prove that (Z, z) is a final coalgebra.