Corecursive Algebras: A Study of General Structured Corecursion (Extended Abstract)

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Abstract. We study general structured corecursion, dualizing the work of Osius, Taylor, and others on general structured recursion. We call an algebra of a functor *corecursive* if it supports general structured corecursion: there is a unique map to it from any coalgebra of the same functor. The concept of *antifounded* algebra is a statement of the bisimulation principle. We show that it is independent from corecursiveness: Neither condition implies the other. Finally, we call an algebra *focusing* if its codomain can be reconstructed by iterating structural refinement. This is the strongest condition and implies all the others.

1 Introduction

A line of research started by Osius and Taylor studies the categorical foundations of general structured recursion. A *recursive coalgebra* (RCA) is a coalgebra of a functor F with a unique coalgebra-to-algebra morphism to any F-algebra. In other words, it is an algebra guaranteeing unique solvability of any structured recursive diagram. The notion was introduced by Osius [Osi74] (it was also of interest to Eppendahl [Epp99]; we studied construction of recursive coalgebras from coalgebras known to be recursive with the help of distributive laws of functors over comonads [CUV06]).

Taylor introduced the notion of wellfounded coalgebra (WFCA) and proved that, in a Set-like category, it is equivalent to RCA [Tay96a,Tay96b],[Tay99, Ch. 6]. Formulated in terms of Jacobs's next-time operator [Jac02], it states that any subset of the carrier of the coalgebra containing its next-time subset is isomorphic to the carrier; in other words, the carrier is the least fixed-point of the next-time operator. As this subset is given by those elements of the carrier whose recursive calls tree is wellfounded, the principle really states that the "wellfounded core" of the coalgebra carrier coincides with the whole carrier [BC05]. A closely related characterization consists in the reconstruction of the coalgebra carrier by iterating the next-time operator on the empty set. Under mild assumptions, it is equivalent to both WFCA and RCA.

Adámek et al. [ALM07] provided extra characterizations for the important case when the functor has an initial algebra. Backhouse and Doornbos [DB96] studied wellfoundedness in a relational setting.

We decided to tackle the dual notions. The importance of this line of research lies in the study of structured recursive definitions which make sense not because of specific properties of the coalgebra marshalling the recursive call arguments but rather thanks to the algebra assembling the recursive call results: here we speak of *general structured corecursion*. This is typical of definitions that work because of "productivity" rather than "termination", e.g., guarded-by-constructors definitions of functions with a coinductive codomain. To our surprise, none of the equivalences established in the dual situation carries over. The duals of the conditions that made those equivalences possible become unreasonable and are false in usual categories, specifically in Set.

The dual of RCA is the notion of *corecursive algebra* (CRA): we call an algebra corecursive if there is a unique map from any coalgebra. Here the first discrepancy arises: while a well-known fact states that initial algebras support primitive recursion and, more generally, a recursive coalgebra is parametrically recursive ([Tay99, Ch. 6]), the dual is not true: corecursiveness with the option of an escape (complete iterativity in the sense of Milius [Mil05]) is a strictly stronger condition than plain corecursiveness.

The dual of WFCA is the notion of *antifounded algebra* (AFA). The dual of the next-time operator maps a quotient of the carrier of an algebra to the quotient identifying the results of applying the algebra structure to elements that were identified in the original quotient. AFA is a categorical formulation of the principle of bisimulation: if a quotient is finer than its next-time quotient, then it must be must be isomorphic to the algebra carrier. Here also the equivalence with CRA fails: both implication turn out to be false for rather simple algebras in Set.

Finally, we call an algebra *focusing* (FA), if it satisfies the dual condition of the one stating that the coalgebra carrier can be reconstructed by iterating the next-time operator on the empty set. In this case, instead of constructing a chain of subsets of the coalgebra carrier, we construct an inverse chain of quotients of the algebra carrier, iterating the dual next-time operator on the quotient by the total relation. Intuitively, each iteration of the dual next-time operator refines the quotient. And while a solution of a recursive diagram in the recursive case is obtained by extending the approximations to larger subsets of the intended domain, now it is obtained by sharpening the approximations to range over finer quotients of the intended codomain. FA turns out to be the strongest of the conditions, it implies both AFA and CRA. The inverse implications turns out to be false.

Throughout the article we are interested in conditions on an algebra (A, α) of an endofunctor F on a category $\mathcal{C}$. We assume that $\mathcal{C}$ has pushouts of epis and that F preserves epis and pushouts of epis. Our prime examples are $\mathcal{C}$ being Set and F a polynomial functor.

2 Corecursive Algebras

Our central object of study in this paper is the notion of corecursive algebra, the dual of Osius's concept recursive coalgebra [Osi74].

Definition 1. *An algebra (A, α) of an endofunctor F on a category $\mathcal{C}$ is called* corecursive *(CRA) if for every coalgebra (C, γ) there exists a unique map $f : C \to A$ (a coalgebra-to-algebra map) making the following diagram commute:*

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & FC \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ A & \xleftarrow[\alpha]{} & FA. \end{array}$$

We write separately CRA-existence and CRA-uniqueness for the statements that the diagram has at least and at most one solution, respectively.

An algebra is corecursive if every structured recursive diagram (= coalgebra-to-algebra map diagram) based on it defines a function (in the sense of turning out to be a definite description). The inverse of the final F-coalgebra, whenever it exists, is trivially a corecursive algebra. However, there are examples of interesting corecursive algebras that arise in different ways.

Example 1. We illustrate the definition with a corecursive algebra in Set, for the functor $FX = E \times X^2$, where E is a fixed set. The carrier is the set of streams over E, $A = \mathsf{Stream}(E)$. The algebra structure is defined as follows:

$$\begin{array}{ll} \alpha : E \times \mathsf{Stream}(E)^2 \to \mathsf{Stream}(E) & \mathsf{merge} : \mathsf{Stream}(E)^2 \to \mathsf{Stream}(E) \\ \alpha(e, s_1, s_2) = e :: \mathsf{merge}(s_1, s_2) & \mathsf{merge}(e :: s_1, s_2) = e :: \mathsf{merge}(s_2, s_1). \end{array}$$

It is easy to see that this algebra is corecursive, although it is not the inverse of the final F-coalgebra, which is the set of non-wellfounded binary trees with nodes labelled by elements of E.

The next notion is an important variation.

Definition 2. *An algebra (A, α) is called* parametrically corecursive *(pCRA) if for every object C and map $\gamma : C \to FC + A$ (that is, coalgebra of $F(-) + A$), there exists a unique map $f : C \to A$ making the following diagram commute:*

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & FC + A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff + \mathsf{id}_A} \\ A & \xleftarrow[{[\alpha, \mathsf{id}_A]}]{} & FA + A. \end{array}$$

This notion is known under the name of *completely iterative algebra* [Mil05].[4] While this term is well-established, we use a different one here for better fit with the topic of this article (the adjective "parametrically" remains idiosyncratic however).

[4] In this terminology, the word "iterative" is used as dual to "general-recursive" in the sense of tail-recursiveness. To be precise, "completely iterative" means that a unique solution exists for every coalgebra while "iterative" refers to the existence of such a solution for every finitary coalgebra, i.e., every coalgebra with a finitary carrier.

To be parametrically corecursive, an algebra must define a function also from diagrams where, for some argument values, the return value of the function is given by an "escape". The inverse of the final coalgebra always has this property [UV99]. Example 1 also satisfies pCRA. We leave the verification to the reader.

Proposition 1. *pCRA ⇒ CRA : A parametrically corecursive coalgebra is corecursive.*

Proof. Given a coalgebra (C, γ), the unique solution of the pCRA diagram for the map $(C, C \xrightarrow{\gamma} FC \xrightarrow{\mathsf{inl}} FC + A)$ is trivially also the unique solution of the CRA diagram for (C, γ). □

The following counterexamples show that the converse is not true (differently from the dual situation of recursive and parametrically recursive coalgebras). We exhibit an algebra that is corecursive but not parametrically corecursive.

Example 2. In the category Set, we use the functor $FX = X \times X$. An interesting observation is that any corecursive algebra (A, α) for this F must have exactly one fixed point, that is, one element a such that $\alpha(a, a) = a$. We take the following algebra structure on the three-element set $A = 3 = \{0, 1, 2\}$:

$$\begin{aligned}&\alpha : 3 \times 3 \to 3\\ &\alpha(1, 2) \ \ = 2\\ &\alpha(n, m) = 0 \quad \text{if } n \neq 1.\end{aligned}$$

Proposition 2. *CRA ⇏ pCRA-uniqueness: Example 2 is corecursive, but does not satisfy the uniqueness property for parametrically corecursive algebras.*

Example 3. Consider the following algebra (A, α) for the functor $FX = X \times X$ in Set: We take A to be $\mathbb{N}$ and define the algebra structure by

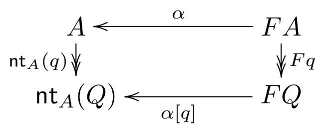
$$\begin{aligned}&\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\\ &\alpha(1, m) = m + 2\\ &\alpha(n, m) = 0 \quad \text{if } n \neq 1.\end{aligned}$$

Proposition 3. *CRA ⇏ pCRA-existence: Example 3 is corecursive, but does not satisfy the existence property for parametrically corecursive algebras.*

3 Antifounded Algebras

Now we turn to the dual of Taylor's wellfounded coalgebras. We follow state the definition with the help of the dual of the next-time operator of Jacobs [Jac02]. Remember that we assume that the category $\mathcal{C}$ has pushouts of epis and that the functor F preserves epis and pushouts of epis.

Definition 3. *Given an algebra* (A, α). *Let* $(Q, q : A \twoheadrightarrow Q)$ *be a quotient of* A *(i.e., an epi with* A *as the domain*[5]*). We define a new quotient* $(\mathsf{nt}_A(Q), \mathsf{nt}_A(q) : A \twoheadrightarrow \mathsf{nt}_A(Q))$ *(the* next-time *quotient) by the following pushout diagram:*



Note that $\mathsf{nt}_A(q)$ is guaranteed to be an epi, as a pushout of an epi.

Notice that we abuse notation (although in a fairly standard fashion): First, nt_A is really parameterized not by the object A, but the algebra (A, α). And further, nt_A operates on a quotient (Q, q) and returns another quotient given by the vertex and one of the side morphisms of the pushout. It is a convention of convenience to denote the vertex by $\mathsf{nt}_A(Q)$ and the morphism by $\mathsf{nt}_A(q)$.

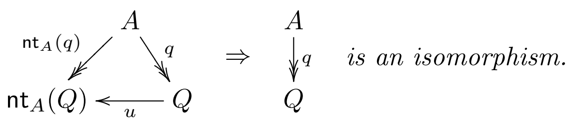
In particular, in the category Set we can give an intuitive definition of nt_A in terms of quotients by equivalence relations. In Set, a quotient is, up to isomorphism, an epi $q : A \twoheadrightarrow A/\equiv$, where $\equiv$ is an equivalence relation on A, with $q(a) = [a]_\equiv$. Its next-time quotient can be represented similarly: $\mathsf{nt}_A(A/\equiv) = A/\equiv'$, where $\equiv'$ is the reflexive-transitive closure of the relation defined by:

$$\forall y_0, y_1 \in FA.\, y_0\ (F\equiv)\ y_1 \Rightarrow \alpha(y_0) \equiv' \alpha(y_1),$$

where $F\equiv$ is the lifting of $\equiv$ to FA: it identifies elements of FA that have the same *shape* and equivalent elements of A in corresponding *positions* (if $\equiv$ is given by a span $(R, r_0, r_1 : R \to A)$, $F\equiv$ is just (FR, Fr_0, Fr_1)).

The following definition is the dual of Taylor's definition of *wellfounded algebra* [Tay96a,Tay96b,Tay99].

Definition 4. *An algebra* (A, α) *is called* antifounded *(AFA) if for every quotient* $(Q, q : A \twoheadrightarrow Q)$, *if* $\mathsf{nt}_A(q)$ *factorizes through* q, *then* q *is an isomorphism. In diagrams:*



Note that, if $\mathsf{nt}_A(q)$ factorizes, i.e., u exists, then it is necessarily unique, as q is an epi. Note also that q being an isomorphism means that id_A factorizes through q, i.e., that q is a split mono.

Example 1 is an antifounded algebra. Indeed, let $q : \mathsf{Stream}(E) \twoheadrightarrow \mathsf{Stream}(E)/\equiv$ be a quotient of $\mathsf{Stream}(E)$ such that $\mathsf{nt}_A(q)$ factors through q. Let $\equiv'$ be the equivalence relation giving the next-time quotient, that is, $\mathsf{nt}_A(\mathsf{Stream}(E)/\equiv) = \mathsf{Stream}(E)/\equiv'$. It is the reflexive-transitive closure of the relation given by:

$$\begin{array}{l}\forall e \in E, s_{00}, s_{01}, s_{10}, s_{11} \in \mathsf{Stream}(E).\\ \quad s_{00} \equiv s_{10} \wedge s_{01} \equiv s_{11} \Rightarrow e :: \mathsf{merge}(s_{00}, s_{01}) \equiv' e :: \mathsf{merge}(s_{10}, s_{11}).\end{array}$$

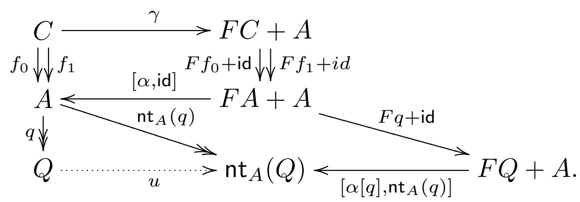
[5] We do not bother to identify equivalent epis, see below.

Notice that this relation is already reflexive and transitive, thus the closure is not necessary. The hypothesis that $\mathsf{nt}_A(q)$ factors through q tells us that $\equiv$ is finer than $\equiv'$, that is, $\forall s_0, s_1 \in \mathsf{Stream}(E).\, s_0 \equiv s_1 \Rightarrow s_0 \equiv' s_1$. We want to prove that $\equiv$ must be equality. In fact, suppose $s_0 \equiv s_1$, then also $s_0 \equiv' s_1$. This means that they must have the same head element e_0 and that their *unmerged* parts must be equivalent: if $s_{00}, s_{01}, s_{10}, s_{11}$ are such that $s_0 = e_0 :: \mathsf{merge}(s_{00}, s_{01})$ and $s_1 = e_0 :: \mathsf{merge}(s_{10}, s_{11})$, then it must be $s_{00} \equiv s_{10}$ and $s_{01} \equiv s_{11}$; repeating the argument for these two equivalences, we can deduce that s_0 and s_1 have the same second and third element, and so on. In conclusion, $s_0 = s_1$ as desired.

Theorem 1. *AFA $\Rightarrow$ pCRA-uniqueness: An antifounded algebra (A, α) satisfies the uniqueness part of the parametric corecursiveness condition.*

Proof. Assume that (A, α) satisfies AFA and let f_0 and f_1 be two solutions of the pCRA diagram for some $(C, \gamma : C \to FC + A)$. We must prove that $f_0 = f_1$.

Let $(Q, q : A \to Q)$ be the coequalizer of f_0 and f_1. As any coequalizer, it is epi. We apply the next-time operator to it. We prove that $\mathsf{nt}_A(q) \circ f_0 = \mathsf{nt}_A(q) \circ f_1$; the proof is summarized by this diagram:



By the fact that f_0 and f_1 are solutions of the pCRA diagram, the top rectangle commutes for both of them. By definition of the nt_A operator, the lower-right parallelogram commutes. Therefore, we have that $\mathsf{nt}_A(q) \circ f_0 = [\alpha[q] \circ F(q \circ f_0), \mathsf{nt}_A(q)] \circ \gamma$ and $\mathsf{nt}_A(q) \circ f_1 = [\alpha[q] \circ F(q \circ f_1), \mathsf{nt}_A(q)] \circ \gamma$. But $q \circ f_0 = q \circ f_1$, because q is the coequalizer of f_0 and f_1, so the right-hand sides of the two previous equalities are the same. We conclude that $\mathsf{nt}_A(q) \circ f_0 = \mathsf{nt}_A(q) \circ f_1$.

Now, using once more that q is the coequalizer of f_0, f_1, there must exist a unique map $u : Q \to \mathsf{nt}_A(Q)$ such that $u \circ q = \mathsf{nt}_A(q)$. By AFA, this implies that q is an isomorphism. As $q \circ f_0 = q \circ f_1$, it follows that $f_0 = f_1$. □

However, AFA does not imply CRA-existence (and therefore, does not imply pCRA-existence), as attested by the following counterexample.

Example 4. In Set, we use the identity functor $FX = X$ and the successor algebra on natural numbers: $A = \mathbb{N}$ and $\alpha : \mathbb{N} \to \mathbb{N}$ is defined by $\alpha(n) = n + 1$.

Proposition 4. *AFA $\not\Rightarrow$ CRA-existence: Example 4 satisfies AFA but not CRA-existence.*

The vice versa also does not hold: CRA does not imply AFA, as evidenced by the following counterexample.

Example 5. We use the functor $FX = 2^* \times X$ in Set, where 2^* is the set of lists of bits (binary words). We construct an F-algebra on the carrier $A = \mathsf{Stream}(2^*)/\simeq$, where $\simeq$ is the equivalence relation defined below. We are particularly interested in streams of a special kind: those made of *incremental* components that stabilize after at most one step. Formally, if $l \in 2^*$ and $i, j \in 2$, we define $\bar{l}^{ij} = (li, lij, lijj, lijjj, \ldots)$, that is,

$$\bar{l}^{00} = (l0, l00, l000, l0000, l00000, \ldots) \quad \bar{l}^{01} = (l0, l01, l011, l0111, l01111, \ldots)$$
$$\bar{l}^{10} = (l1, l10, l100, l1000, l10000, \ldots) \quad \bar{l}^{11} = (l1, l11, l111, l1111, l11111, \ldots).$$

The relation $\simeq$ is the least congruence such that $\bar{l}^{01} \simeq \bar{l}^{10}$ for every l. This means that two streams that begin in the same way but then stabilize in one of the two forms above will be equal: $(l_0, \ldots, l_{k-1}) +\!\!+\, \bar{l}^{01} \simeq (l_0, \ldots, l_{k-1}) +\!\!+\, \bar{l}^{10}$. In other words, the equivalence classes of $\simeq$ are $\{(l_0, \ldots, l_{k-1}) +\!\!+\, \bar{l}^{01}, (l_0, \ldots, l_{k-1}) +\!\!+\, \bar{l}^{10}\}$ for elements in one of those two forms, and singletons for elements not in those forms. Notice that we do not equate elements of the forms $\bar{l}^{00}$ and $\bar{l}^{11}$. For simplicity, we will write elements of A just as sequences, in place of equivalence classes. So if $s \in \mathsf{Stream}(2^*)$, we will use s also to indicate $[s]_{\simeq}$. We leave it to the reader to check that all our definitions are invariant with respect to $\simeq$. We now define an algebra structure α on this carrier by:

$$\alpha : 2^* \times (\mathsf{Stream}(2^*)/\simeq) \to \mathsf{Stream}(2^*)/\simeq$$
$$\alpha(l, s) = l :: s.$$

Proposition 5. *pCRA $\not\Rightarrow$ AFA: Example 5 satisfies pCRA but not AFA.*

Proof. First we prove that Example 5 satisfies pCRA. Given some $(C, \gamma : C \to 2^* \times C + A)$, we want to prove that there is a unique solution to pCRA diagram. We already know that there is a unique map from C to the inverse $(::)$ of the final coalgebra $(\mathsf{Stream}(2^*), \langle \mathsf{head}, \mathsf{tail} \rangle)$ of the functor $FX = 2^* \times X$. Notice that A is just a quotient of $\mathsf{Stream}(2^*)$ and that α operates on A in the same way that $(::)$ operates on $\mathsf{Stream}(2^*)$. Therefore, the unique solution for $(::)$ is also a unique solution for α (after quotienting).

Now we prove that Example 5 does not satisfy AFA. With this goal we define an equivalence relation $\equiv$ on $A = \mathsf{Stream}(2^*)/\simeq$ such that $\mathsf{nt}_A(A/\equiv)$ factorizes through $A/\equiv$ but $\equiv$ is strictly coarser than $\simeq$. The relation $\equiv$ is the reflexive closure of the following: $\forall l \in 2^*, i_0, i_1, j_0, j_1 \in 2.\, \bar{l}^{i_0 j_0} \equiv \bar{l}^{i_1 j_1}$. In other words, $\equiv$ identifies all elements in the form $\bar{l}^{ij}$ that have the same base sequence l. Contrary to the case of $\simeq$, we do not extend $\equiv$ to a congruence. So if s_0 is not in the form $\bar{l}^{ij}$, then $s_0 \equiv s_1$ is true only if $s_0 \simeq s_1$. This equivalence relation is clearly coarser than $\simeq$, since $\bar{l}^{00} \equiv \bar{l}^{11}$ but $\bar{l}^{00} \not\simeq \bar{l}^{11}$.

Let $\equiv'$ be the next-time equivalence relation of $\equiv$, i.e., such that $\mathsf{nt}_A(A/\equiv) = A/\equiv'$. Concretely, $\equiv'$ is the relation (already reflexive and transitive) defined by:

$$\forall l \in 2^*, s_0, s_1 \in \mathsf{Stream}(2^*).\ s_0 \equiv s_1 \Rightarrow l :: s_0 \equiv' l :: s_1.$$

We prove that $\equiv$ is finer than $\equiv'$, i.e., if $s_0 \equiv s_1$, then $s_0 \equiv' s_1$. There are two cases.

If s_0 or s_1 is not in the form $\bar{l}^{ij}$, then its equivalence class is a singleton by definition, so the other element must be equal to it and the conclusion follows by reflexivity.

If both s_0 and s_1 are in the form $\bar{l}^{ij}$, then their base element must be the same l, by definition of $\equiv$. There are four cases for each of the two elements, according to what their i and j parameters are. By considerations of symmetry and reflexivity, we can reduce the cases to just two:

- $s_0 = \bar{l}^{00}$ and $s_1 = \bar{l}^{01}$: We can write the two elements alternatively as $s_0 = l0 : \overline{l0}^{00}$ and $s_1 = l0 : \overline{l0}^{11}$; since $\overline{l0}^{00} \equiv \overline{l0}^{11}$, we conclude that $s_0 \equiv' s_1$;
- $s_0 = \bar{l}^{00}$ and $s_1 = \bar{l}^{11}$: By the previous case and its dual we have $s_0 \equiv' \bar{l}^{01}$ and $s_1 \equiv' \bar{l}^{10}$; but $\bar{l}^{01} \simeq \bar{l}^{10}$ so $s_0 \equiv' s_1$ by transitivity. □

We now turn to a higher-level view of antifounded algebras. This is in terms of the classical fixed point theory for preorders and monotone endofunctions.

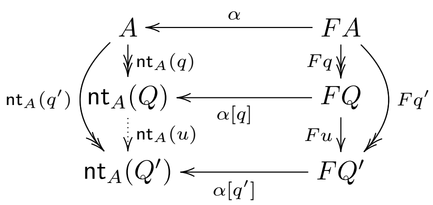
For a (locally small) category $\mathcal{C}$ and an object A, we define the category of quotients of A, called $\mathbf{Quo}(A)$ as follows:

- an object is an epimorphism $(Q, q : A \twoheadrightarrow Q)$,
- a map between (Q, q), (Q', q') is a map $u : Q \to Q'$ such that $u \circ q = q'$.

Clearly there can be at most one map between any two objects, so this category is a preordered set. (In the standard definition of the category, equivalent epis are identified, so it becomes a poset. We have chosen to be content with a preorder; the cost is that universal properties define objects up to isomorphism.) We tend to write $Q \leq Q'$ instead of $u : (Q, q) \to (Q', q')$, leaving q, q' and the unique u implicit.

Clearly, $\mathbf{Quo}(A)$ has (A, id_A) as the initial and $(1, !_A)$ as the final object.

Now, nt_A sends objects of $\mathbf{Quo}(A)$ to objects of $\mathbf{Quo}(A)$. It turns out that it can be extended to also act on maps. For a map $u : (Q, q) \to (Q', q')$, we define $\mathsf{nt}_A(u) : (\mathsf{nt}_A(Q), \mathsf{nt}_A(q)) \to (\mathsf{nt}_A(Q'), \mathsf{nt}_A(q'))$ as the unique map from a pushout, as shown in the following diagram:



Given that $\mathbf{Quo}(A)$ is a preorder, this makes nt_A trivially a functor (preservation of the identities and composition is trivial). In preorder-theoretic terms, we say that nt_A is a monotone function.

We can notice that (A, id_A) is trivially a fixed point of nt_A. Since it is the least element of $\mathbf{Quo}(A)$, it is the least fixed point.

The condition of (A, α) being antifounded literally says that, for any Q, $Q \leq \mathsf{nt}_A(Q)$ implies $Q \leq A$, i.e., that A is an upper bound on the post-fixed points of nt_A. Taking into account that A, by being the least element, is also trivially a post-fixed point, this amounts to A being the greatest post-fixed point. Fixed point theory (or, if you wish, Lambek's lemma) tells us that the greatest post-fixed point is also the greatest fixed point.

So, in fact, (A, α) being antifounded means that (A, id_A) is a unique fixed point of nt_A. (Recall that this is up to isomorphism.)

4 Focusing Algebras

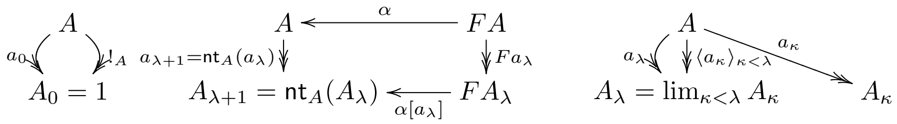
We iterate the next-time operator, starting with the final quotient. At transfinite iterations, we are not guaranteed that we still obtain a quotient: In Proposition 7 we will prove that, for Example 4, the iteration at stage ω is not a quotient anymore. In the rest of the paper, when we talk about such iterations at limit ordinals, we implicitly require that the limit (together with its previous stages) is a quotient itself.

Definition 5. *Given an algebra (A, α), for any ordinal λ we partially-define (A_λ, a_λ) (the λ-th iteration of nt_A on the final object $(1, !_A)$ of $\mathbf{Quo}(A)$) and maps $p_\lambda : A_{\lambda+1} \to A_\lambda$, $p_{\lambda,\kappa} : A_\lambda \to A_\kappa$ (for λ a limit ordinal and $\kappa < \lambda$) in $\mathcal{C}$ by simultaneous recursion by*

$$\begin{array}{llll}
A_0 = 1 & a_0 = !_A & p_0 = !_{A_1} & \\
A_{\lambda+1} = \mathsf{nt}_A(A_\lambda) & a_{\lambda+1} = \mathsf{nt}_A(a_\lambda) & p_{\lambda+1} = \mathsf{nt}_A(p_\lambda) & \\
A_\lambda = \lim_{\kappa<\lambda} A_\kappa & a_\lambda = \langle a_\kappa \rangle_{\kappa<\lambda} & p_\lambda = \langle p_\kappa \circ \mathsf{nt}_A(p_{\lambda,\kappa}) \rangle_{\kappa<\lambda} & \textit{if } \lambda \textit{ is a lim. ord.} \\
 & & p_{\lambda,\kappa} = \pi_{\lambda,\kappa} & \textit{if } \kappa < \lambda
\end{array}$$

if the limit exists and $\langle a_\kappa \rangle_{\kappa<\lambda}$ is epi; otherwise undefined

Diagrammatically,



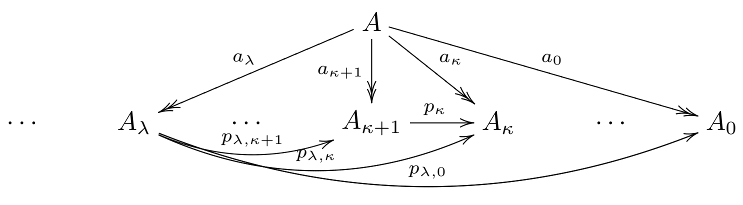
The limit in the limit ordinal case is of the following diagram in $\mathcal{C}$:

$$(A_\kappa, p_\kappa, p_{\kappa,\iota}\ (\kappa \text{ lim. ord.}, \iota < \kappa))_{\kappa<\lambda} < \kappa))_{\kappa<\lambda}.$$

Lemma 1. *The above definition is well-formed: for any λ,*

- *a_λ is an epi (so, for any λ, nt_A is applicable to (A_λ, a_λ), ensuring $(A_{\lambda+1}, a_{\lambda+1})$ is defined),*
- *$p_\lambda \circ a_{\lambda+1} = a_\lambda$ and $p_{\lambda,\kappa} \circ a_\lambda = a_\kappa$ (if λ is a limit ordinal, $\kappa < \lambda$) (so, for any λ, $(A, (a_\kappa)_{\kappa<\lambda})$ in the definition of a_λ for λ a limit ordinal form a cone, ensuring a_λ is defined)*

Diagrammatically,



It is very important to realize that we only accept $\lim_{\kappa<\lambda} A_\kappa$ (which is a limit in $\mathcal{C}$) as A_λ for λ a limit ordinal, if it is a quotient of A. This is by no means granted. As the next proposition shows, this implies that A_λ is also a limit in $\mathbf{Quo}(A)$, but the vice versa need not be true. The carrier of a limit in $\mathbf{Quo}(A)$ is not necessarily a limit in $\mathcal{C}$, as evidenced by our analysis of Example 4 below.

Proposition 6. *If A_λ is defined for a limit ordinal (meaning that $(A_\lambda, (p_{\lambda,\kappa})_{\kappa<\lambda})$ is a limiting cone in $\mathcal{C}$ and $a_\lambda = \langle a_\kappa \rangle_{\kappa<\lambda}$ is epi), then $((A_\lambda, a_\lambda), (p_{\lambda,\kappa})_{\kappa<\lambda})$ is a limiting cone in $\mathbf{Quo}(A)$.*

Given that $\mathbf{Quo}(A)$ is a preorder, we have learned that $(A_\kappa)_{\kappa<\lambda}$ is an inverse chain (if all A_κ are defined) and the limit is the infimum.

Lemma 2. *If A_λ is defined and $A_\lambda \le A_{\lambda+1}$, then A_λ is the greatest fixed point of nt_A.*

Definition 6. *(A, α) is* λ-focusing *(λ-FA) if A_λ is defined and $A_\lambda \cong A$.*

We show that Example 1 is ω-focusing. In fact we claim that, in this case, $A_n = \mathsf{Stream}(E)/{\equiv_n}$, where $\equiv_n$ is the equivalence relation defined by $s_0 \equiv_n s_1$ if the first $2^n - 1$ elements of s_0 and s_1 are the same. The claim is clearly true for $n = 0$, because $\equiv_0$ is the total relation. Assume, by induction hypothesis, that $A_n = \mathsf{Stream}(E)/{\equiv_n}$. Then $A_{n+1} = \mathsf{nt}_A(\mathsf{Stream}(E)/{\equiv_n}) = \mathsf{Stream}(E)/{\equiv'_n}$. Now $s_0 \equiv'_n s_1$ holds if $s_0 = e_0 :: \mathsf{merge}(s_{00}, s_{01})$ and $s_1 = e_0 :: \mathsf{merge}(s_{10}, s_{11})$ with $s_{00} \equiv_n s_{10}$ and $s_{01} \equiv_n s_{11}$. By induction hypothesis, this means that the first $2^n - 1$ elements of s_{00} and s_{10} are the same and the first $2^n - 1$ elements of s_{01} and s_{11} are also the same. In conclusion, the first $1+(2^n-1)+(2^n-1) = 2^{n+1}-1$ elements of s_0 and s_1 are the same, that is $s_0 \equiv_{n+1} s_1$, as claimed.

We proved that A_n is isomorphic to the set E^{2^n-1} of vectors of length 2^n-1, with p_n the projection giving the first $2^n - 1$ elements of a vector of length $2^{n+1} - 1$. Standard reasoning shows that the $\lim_{i<\omega} A$ is $\mathsf{Stream}(E)$.

There are examples of λ-focusing algebras that do not converge at the first limit ordinal ω but at later stages. Here is an example that converges at 2ω.

Example 6. Let us use the functor $FX = X + \mathbb{N} \times X$ in Set. We define an F-algebra with carrier $A = 2\omega + 1 = \{0, 1, \ldots, \omega, \omega+1, \omega+2, \ldots, 2\omega\}$:

$$\begin{aligned}
&\alpha : (2\omega+1) + \mathbb{N} \times (2\omega+1) \to 2\omega+1 \\
&\alpha(\mathsf{inl}(x)) \quad\;\; = x + 1 \\
&\alpha(\mathsf{inr}(n, x)) = \min(\omega + x - n, 2\omega).
\end{aligned}$$

Theorem 2. *λ-FA $\Rightarrow$ AFA: If an algebra (A, α) is λ-focusing, it is antifounded.*

Proof. Assume that (A, α) is λ-focusing, i.e., that A_λ is defined and $A_\lambda \cong A$. Then $A_\lambda \cong A \le A_{\lambda+1}$ trivially, as A is the least element in the preorder $\mathbf{Quo}(A)$. It follows by the previous lemma that A_λ, which is isomorphic to A, is the greatest fixed point of nt_A, i.e., that (A, α) is antifounded. □

The vice versa does not hold: Some antifounded algebras are not focusing.

Proposition 7. *AFA $\not\Rightarrow \exists\lambda.\,\lambda$-FA: Ex. 4 satisfies AFA but not λ-FA for any λ.*

Notice that $(A_i)_{i<\omega}$ has the limit $\mathbb{N}$ in $\mathbf{Quo}(A)$. So we have to be mindful of the subtle distinction: λ-FA states that the limit exists in $\mathcal{C}$ and happens to be a quotient; this is a strictly stronger requirement than the condition that the limit exists in $\mathbf{Quo}(A)$.

Theorem 3. *λ-FA $\Rightarrow$ pCRA: If an algebra (A, α) is λ-focusing, it is parametrically recursive.*

The proof uses the inverse chain $(A_\kappa)_{\kappa<\lambda+1}$ as the sequence of codomains for *fuzzy* approximations of the solution. The fact that $A = A_\lambda$ is the inverse limit establishes that a (sharp) function is achieved. This is analogous to the dual situation where a (total) solution arises from a sequence of *partial* approximations defined on a chain of subsets of the given domain to which the chain is required to have as the direct limit.

Proof (Sketch, full proof in the Appendix). Assume that (A, α) is λ-focusing. Given $(C, \gamma : C \to FC + A)$, we define, for any κ, a map $f_\kappa : C \to A_\kappa$ by

$$\begin{aligned} f_0 &= !_C \\ f_{\kappa+1} &= [\alpha[a_\kappa] \circ Ff_\kappa, a_{\kappa+1}] \circ \gamma \\ f_\kappa &= \langle f_\iota \rangle_{\iota<\kappa} && \text{if } \kappa \text{ is a lim. ord.} \end{aligned}$$

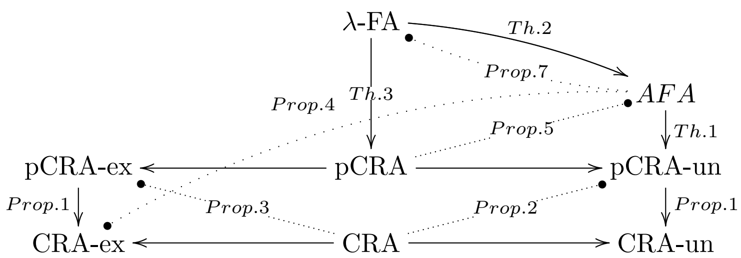
The map $f = f_\lambda$ is the unique solution. □

Finally, notice that since pCRA does not imply AFA, it cannot imply λ-FA. Example 5 shows this: We proved that it satisfies pCRA but not AFA, therefore it also doesn't satisfy λ-FA.

5 Conclusion

We have looked at some notions of support for general structured corecursion/coinduction. They are all properties on an algebra (A, α) of a fixed functor F. The conditions CRA/pCRA state that we can uniquely solve all structured recursive diagrams based on (A, α). The condition AFA asserts that the principle of bisimulation holds for the carrier A: Every equivalence on A that is finer than its own structural refinement must be equality. Finally, λ-FA says that we can reconstruct A by iterating structural refinement.

The relations between the four conditions CRA, pCRA, AFA, and λ-FA are summarized by the following diagram. The solid lines indicate implications, the dotted lines indicate non-implications.



It is clear from this study that the general structured recursion/induction is much richer than general structured corecursion/coinduction from which we drew inspiration. In particular, for Set-like categories, straightfoward dualization of the different equivalent conditions of recursiveness leads to inequivalent conditions of corecursiveness. There is still much work to do in this area of investigation. We must understand fully each of the defined conditions to grasp their significance and to seek variants that are more in line with our intuitive grasp. We expect that this enquiry will produce new and exciting results.

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THE FOLLOWING IS AN APPENDIX WITH THE OMITTED PROOFS.

A Omitted Proofs

Properties of Example 2 (Prop.2)

(CRA) Example 2 satisfies CRA. Let (C, γ) be a coalgebra. We prove that the only possible solution f of the CRA diagram is the constant 0. In fact, for $c \in C$, it cannot be $f(c) = 1$, because 1 is not in the range of α. On the other hand, if $f(c) = 2$, then we must have $f(c) = \alpha((f \times f)(\gamma(c)))$. Let us call c_0 and c_1 the two components of $\gamma(c)$: $\gamma(c) = (c_0, c_1)$. Then we have $f(c) = \alpha(f(c_0), f(c_1))$. For $f(c)$ to be equal to 2, it is necessary that $f(c_0) = 1$ and $f(c_1) = 2$. But we already determined that $f(c_0) = 1$ is impossible. In conclusion, there is a unique solution: $f(c) = 0$ for every $c \in C$.

(not pCRA-uniqueness) Example 2 does not satisfy pCRA-uniqueness. Specifically, the uniqueness property fails, as shown by the following two solutions of the pCRA diagram for $C = \mathbb{B}$ and $\gamma : \mathbb{B} \to \mathbb{B} \times \mathbb{B} + 3$ defined by $\gamma(\mathsf{true}) = \mathsf{inr}(1), \gamma(\mathsf{false}) = \mathsf{inl}(\mathsf{true}, \mathsf{false})$:

$$
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{\gamma} & \mathbb{B} \times \mathbb{B} + 3 \\
{\scriptstyle f_0}\downarrow\downarrow{\scriptstyle f_1} & & {\scriptstyle f_0 \times f_0 + \mathsf{id}}\downarrow\downarrow{\scriptstyle f_1 \times f_1 + \mathsf{id}} \\
3 & \xleftarrow[{[\alpha, \mathsf{id}]}]{} & 3 \times 3 + 3
\end{array}
\qquad
\begin{array}{ll}
f_0(\mathsf{true}) = 1 & f_1(\mathsf{true}) = 1 \\
f_0(\mathsf{false}) = 0 & f_1(\mathsf{false}) = 2.
\end{array}
$$

(pCRA-existence) Example 2 satisfies pCRA-existence: to construct a solution, put it equal to 0 on all argument values on which it is not recursively forced.

(not λ-FA) Example 2 does not satisfy λ-FA for any λ. In fact, we have that $A_0 = 1$, $A_1 = \{\{1\}, \{0, 2\}\}$ and this is already a fixed point of nt_A. Thus $A_\lambda = A_1$ for all $\lambda > 0$. In cases like this, when some A_λ is a fixed point of nt_A, but fails to be isomorphic to A, the algebra (A, α) is not focusing, however the quotient algebra $(A_\lambda, \alpha[a_\lambda])$ focuses. This is analogous to the wellfounded core of a coalgebra in the dual situation.

Properties of Example 3 (Prop.3)

(CRA) Example 3 satisfies CRA, essentially by the same arguments as for Example 2: the unique solution is forced to be the constant 0.

(not pCRA-existence) Example 3 does not satisfy pCRA-existence. Take $C = \mathbb{B}$ and define $\gamma : \mathbb{B} \to \mathbb{B} \times \mathbb{B} + \mathbb{N}$ by $\gamma(\mathsf{true}) = \mathsf{inr}(1)$ and $\gamma(\mathsf{false}) = \mathsf{inl}(\mathsf{true}, \mathsf{false})$. For this case, there is no solution to pCRA diagram. Indeed, a solution should surely satisfy $f(\mathsf{true}) = 1$. Therefore we should also have $f(\mathsf{false}) = \alpha(f(\mathsf{true}), f(\mathsf{false})) = \alpha(1, f(\mathsf{false})) = f(\mathsf{false}) + 2$, which is clearly impossible.

(AFA) Example 3 satisfies AFA. Let $q : \mathbb{N} \twoheadrightarrow \mathbb{N}/\equiv$ be a quotient of $\mathbb{N}$ such that $\mathsf{nt}_A(q)$ factorizes through q. In this case $\mathsf{nt}_A(\mathbb{N}/\equiv) = A/\equiv'$ with $\equiv'$ the reflexive-transitive closure of the relation:

$$\forall n_0, n_1, m_0, m_1 \in \mathbb{N}.\, n_0 \equiv n_1 \wedge m_0 \equiv m_1 \Rightarrow \alpha(n_0, m_0) \equiv' \alpha(n_1, m_1).$$

Saying that $\mathsf{nt}_A(q)$ factors through q is the same as saying that $\forall \in \mathbb{N}.\ [x]_\equiv \subseteq [x]_{\equiv'}$. We want to prove that, under this assumption, $\equiv$ must be equality, that is, every equivalence class is a singleton. So we prove, by induction on x, that $[x]_\equiv = \{x\}$.

- $[1]_\equiv = \{1\}$ because 1 is not in the range of α. Since it is not in the form $\alpha(y)$, it cannot be proved equivalent to any other number according to $\equiv'$. Since $[1]_\equiv \subseteq [1]_{\equiv'}$, the conclusion follows.
- $[0]_\equiv = \{0\}$: Assume $0 \equiv x$ for some other element x. By assumption it is also $0 \equiv' x$. Then it must be $0 = \alpha(n_0, m_0)$ and $x = \alpha(n_1, m_1)$ with $n_0 \equiv n_1$ and $m_0 \equiv m_1$ (transitivity could have been used, but there must always be a first element x that is equivalent by the base rule). But α can produce 0 as result only if the first component of the argument is different from 1, so $n_0 \neq 1$. Consequently, it cannot be that $n_1 = 1$, by the previous case, so $n_1 \neq 1$. In conclusion, $x = \alpha(n_1, m_1) = 0$, as desired.
- Now assume, by induction hypothesis, that $[x]_\equiv = \{x\}$. We shall prove that $[x+2]_\equiv = \{x+2\}$. Assume that $x + 2 \equiv x'$ for some other element x'. By assumption, we also have that $x+2 \equiv' x'$. Then it must be $x+2 = \alpha(n_0, m_0)$ and $x' = \alpha(n_1, m_1)$ with $n_0 \equiv n_1$ and $m_0 \equiv m_1$. By the definition of α, it must necessarily be that $n_0 = 1$ and $m_0 = x$. Then, by the first case above, $n_1 = 1$, and, by induction hypothesis, $m_1 = x$. We conclude that $x' = x + 2$, as desired.

Properties of Example 4 (Prop.4 and 7)

(AFA) Example 4 satisfies AFA. Let $q : A \to A/\equiv$ be a quotient of A such that $\mathsf{nt}_A(q)$ factorizes through q. Note that the definition of $\equiv'$ (the next-time equivalence relation of $\equiv$) is particularly simple, just the reflexive closure of:

$$\forall m_0, m_1 \in \mathbb{N}.\, m_0 \equiv m_1 \Rightarrow m_0 + 1 \equiv' m_1 + 1.$$

So two distinct numbers are equivalent according to $\equiv'$ if and only if they are the successors of elements that are equal according to $\equiv$. There is no need of a transitive closure in this case, since the relation is already transitive. By assumption $\equiv$ is finer than $\equiv'$, that is $\forall m_1, m_2 \in \mathbb{N}.\, m_0 \equiv m_1 \Rightarrow m_0 \equiv' m_1$. We want to prove that $\equiv$ is equality. We prove, by induction on m, that $[m]_\equiv = \{m\}$, that is, every equivalence class is a singleton:

- For $m = 0$ the statement is trivial: $0 \equiv m'$ implies, by hypothesis, that $0 \equiv' m'$, but since 0 is not a successor, this can happen only by reflexivity, that is, if $m' = 0$;

– Assume that $[m]_\equiv = \{m\}$ by induction hypothesis; we must prove that $[m+1]_\equiv = \{m+1\}$; if $m+1 \equiv m'$, then $m+1 \equiv' m'$, which can happen only if $m' \neq 0$ and $m \equiv m'-1$; by induction hypothesis, this implies that $m'-1 = m$, so $m' = m+1$.

(not CRA-existence) Example 4 does not satisfy CRA-existence. Indeed, if we take the trivial coalgebra $(1 = \{0\}, \mathsf{id} : 1 \to 1)$, we see that a solution of the CRA diagram would require $f(0) = f(0)+1$, which is impossible.

(not λ-FA) Example 4 is not focusing. In fact, we have the following sequence of iterations of nt_A:

$$\begin{array}{l} A_0 = \{\bot\}, \quad A_1 = \{0,\bot\}, \quad A_2 = \{0,1,\bot\}, \quad \ldots, \\ A_i = \{0,\ldots,i-1,\bot\}, \quad \ldots, \quad \lim_{i<\omega} A_i = \mathbb{N} \cup \{\bot\}. \end{array}$$

At the limit, the element $\bot$ is not an equivalence class of natural numbers anymore and the limit $\lim_{i<\omega} A_i$ is not a quotient of $A = \mathbb{N}$. So, in this case, the limit exists in Set, but is not a limit in the quotient category $\mathbf{Quo}(A)$. The reason that this happens is that, in Set, the limit of an inverse chain of quotients given by equivalence relations is not necessarily the quotient given by the intersection of the equivalence relations in the inverse chain.

Example 6 is 2ω-focusing

For clarity, we write just n for an equivalence class containing the singleton n, and $\bot$ for the equivalence class containing all the elements not already in some other class. The finite stages of the iterations of nt_A give the following quotients:

$$A_0 = \{\bot\}, \quad A_1 = \{0,\bot\}, \quad A_2 = \{0,1,\bot\}, \quad \ldots, \quad A_i = \{0,\ldots,i-1,\bot\}.$$

In fact, a finite element $i < \omega$ can be written as results of α only in the form $i = \alpha(\mathsf{inl}(i-1))$, which easily shows, by induction, that its equivalence class becomes a singleton exactly at stage $i+1$. On the other hand, all the transfinite elements must be equivalent at stage i because $\omega = \alpha(\mathsf{inr}(i,i)$ and $\omega + n = \alpha(\mathsf{inr}(i+n,i))$ with i and $i+n$ equivalent at stage i. The transfinite stages of iteration of nt_A proceed as follows:

$$\begin{array}{c} A_\omega = \mathbb{N} \cup \{\bot\}, \quad A_{\omega+1} = \mathbb{N} \cup \{\omega,\bot\}, \quad A_{\omega+2} = \mathbb{N} \cup \{\omega,\omega+1,\bot\}, \\ \ldots, \quad A_i = \mathbb{N} \cup \{\omega,\ldots,\omega+i-1,\bot\}, \quad \ldots, \quad A_{2\omega} = 2\omega+1, \end{array}$$

where we identified $\bot$ with 2ω at the last stage.

Proof of Lemma 1

By induction. It is trivial that $p_0 \circ a_1 = !_{A_1} \circ a_1 = !_A = a_0$.

$p_{\lambda+1} \circ a_{\lambda+2} = \mathsf{nt}_A(p_\lambda) \circ \mathsf{nt}_A(a_{\lambda+1}) = \mathsf{nt}_A(a_\lambda) = a_{\lambda+1}$ holds by the induction hypothesis $p_\lambda \circ a_{\lambda+1} = a_\lambda$, implying $\mathsf{nt}_A(p_\lambda) \circ \mathsf{nt}_A(a_{\lambda+1}) = \mathsf{nt}_A(a_\lambda)$ by the definition of the functorial extension of nt_A.

For λ a limit ordinal, $p_\lambda \circ a_{\lambda+1} = \langle p_\kappa \circ \mathsf{nt}_A(p_{\lambda,\kappa})\rangle_{\kappa<\lambda} \circ \mathsf{nt}_A(a_\lambda) = \langle p_\kappa \circ \mathsf{nt}_A(p_{\lambda,\kappa}) \circ \mathsf{nt}_A(a_\kappa)\rangle_{\kappa<\lambda} = \langle p_\kappa \circ \mathsf{nt}_A(a_\kappa)\rangle_{\kappa<\lambda} = \langle p_\kappa \circ a_{\kappa+1}\rangle_{\kappa<\lambda} = \langle a_\kappa\rangle_{\kappa<\lambda} = a_\lambda$, from the induction hypotheses $p_{\lambda,\kappa} \circ a_\lambda = a_\kappa$, implying $\mathsf{nt}_A(p_{\lambda,\kappa}) \circ \mathsf{nt}_A(a_\lambda) = \mathsf{nt}_A(a_\kappa)$ by the definition of the functorial extension of nt_A, and from the induction hypotheses $p_\kappa \circ a_{\kappa+1} = a_\kappa$.

For λ a limit ordinal and $\kappa < \lambda$, $p_{\lambda,\kappa} \circ a_\lambda = \pi_{\lambda,\kappa} \circ \langle a_\kappa\rangle_{\kappa<\lambda} = a_\kappa$. □

Proof of Proposition 6

To see that that

$$((A_\kappa, a_\kappa), p_\kappa, p_{\kappa,\iota}\ (\kappa \text{ lim. ord.}, \iota < \kappa))_{\kappa<\lambda}$$

is a diagram in $\mathbf{Quo}(A)$ we need that $p_\kappa \circ a_{\kappa+1} = a_\iota$ and $p_{\kappa,\iota} \circ a_\kappa = a_\iota$ (κ a limit ordinal, $\iota < \kappa$) for $\kappa < \lambda$. To see that $((A_\lambda, a_\lambda), (p_{\lambda,\kappa})_{\kappa<\lambda})$ is a cone we also need $p_{\lambda,\kappa} \circ a_\lambda = a_\kappa$. But we have proved these equalities already.

To see that $((A_\lambda, a_\lambda), (p_{\lambda,\kappa})_{\kappa<\lambda})$ is a limiting cone, we observe that the sole map to it from a cone $((Q, q), (f_{\lambda,\kappa})_{\kappa<\lambda})$ in $\mathbf{Quo}(A)$ is given by the unique map from $(Q, (f_{\lambda,\kappa})_{\kappa<\lambda})$ to $(A_\lambda, (p_{\lambda,\kappa})_{\kappa<\lambda})$ in $\mathcal{C}$. □

Proof of Lemma 2

This is standard fixed point theory for preorders. A_λ is a post-fixed point of nt_A, as $A_\lambda \leq A_{\lambda+1} = \mathsf{nt}_A(A_\lambda)$. And by induction one checks that $Q \leq A_\kappa$ holds from any post-fixed point of nt_A and any κ: $Q \leq 1 = A_0$ is trivial; $Q \leq \mathsf{nt}_A(Q) \leq \mathsf{nt}_A(A_\kappa) = A_{\kappa+1}$ follows from the induction hypothesis $Q \leq A_\kappa$, as nt_A is monotone; and, finally, $Q \leq \inf_{\iota<\kappa} A_\iota$ is immediate from the induction hypotheses $Q \leq A_\iota$ $(\iota < \kappa)$. □

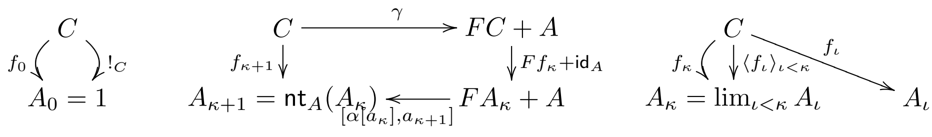
Proof of Theorem 3

Assume that (A, α) is λ-focusing, i.e., that A_λ is defined and $A_\lambda = A_{\lambda+1} = A$ (again we ignore that in general we have isomorphisms, not equalities).

Given $(C, \gamma : C \to FC + A)$, we define, for any κ, a map $f_\kappa : C \to A_\kappa$ by

$$\begin{aligned} f_0 &= !_C \\ f_{\kappa+1} &= [\alpha[a_\kappa] \circ F f_\kappa, a_{\kappa+1}] \circ \gamma \\ f_\kappa &= \langle f_\iota\rangle_{\iota<\kappa} && \text{if } \kappa \text{ is a lim. ord.} \end{aligned}$$

Diagrammatically,



Simultaneously, we show that $p_\kappa \circ f_{\kappa+1} = f_\kappa$ and $p_{\kappa,\iota} \circ f_\kappa = f_\iota$.

$p_0 \circ f_1 = !_{A_1} \circ f_1 = !_C = f_0$ holds trivially.

$p_{\kappa+1} \circ f_{\kappa+2} = \mathsf{nt}_A(p_\kappa) \circ [\alpha[a_{\kappa+1}] \circ Ff_{\kappa+1}, a_{\kappa+2}] \circ \gamma = [\mathsf{nt}_A(p_\kappa) \circ \alpha[a_{\kappa+1}] \circ Ff_{\kappa+1}, \mathsf{nt}_A(p_\kappa) \circ \mathsf{nt}_A(a_{\kappa+1})] \circ \gamma = [\alpha[a_\kappa] \circ F(p_\kappa \circ f_{\kappa+1}), \mathsf{nt}_A(a_\kappa)] \circ \gamma = [\alpha[a_\kappa] \circ Ff_\kappa, a_{\kappa+1}] \circ \gamma = f_{\kappa+1}$ follows from the induction hypothesis $p_\kappa \circ f_{\kappa+1} = f_\kappa$, using the fact $p_\kappa \circ a_{\kappa+1} = a_\kappa$, which implies $\mathsf{nt}_A(p_\kappa) \circ \alpha[a_{\kappa+1}] = \alpha[a_\kappa] \circ Fp_\kappa$ and $\mathsf{nt}_A(p_\kappa) \circ \mathsf{nt}_A(a_{\kappa+1}) = \mathsf{nt}_A(a_\kappa)$ by the definition of the functorial extension of nt_A.

For κ a limit ordinal, $p_\kappa \circ f_{\kappa+1} = \langle p_\iota \circ \mathsf{nt}_A(p_{\kappa,\iota})\rangle_{\iota<\kappa} \circ [\alpha[a_\kappa] \circ Ff_\kappa, a_{\kappa+1}] \circ \gamma = \langle p_\iota \circ \mathsf{nt}_A(p_{\kappa,\iota}) \circ [\alpha[a_\kappa] \circ Ff_\kappa, a_{\kappa+1}] \circ \gamma\rangle_{\iota<\kappa} = \langle p_\iota \circ [\mathsf{nt}_A(p_{\kappa,\iota}) \circ \alpha[a_\kappa] \circ Ff_\kappa, \mathsf{nt}_A(p_{\kappa,\iota}) \circ \mathsf{nt}_A(a_\kappa)] \circ \gamma\rangle_{\iota<\kappa} = \langle p_\iota \circ [\alpha[a_\iota] \circ F(p_{\kappa,\iota} \circ f_\iota), \mathsf{nt}_A(a_\iota)] \circ \gamma\rangle_{\iota<\kappa} = \langle p_\iota \circ [\alpha[a_\iota] \circ Ff_\iota, a_{\iota+1}] \circ \gamma\rangle_{\iota<\kappa} = \langle p_\iota \circ f_{\iota+1}\rangle_{\iota<\kappa} = \langle f_\iota\rangle_{\iota<\kappa} = f_\kappa$ follows from the induction hypotheses $p_{\kappa,\iota} \circ f_\kappa = f_\iota$ and $p_\iota \circ f_{\iota+1} = f_\iota$, using the facts $p_{\kappa,\iota} \circ a_\kappa = a_\iota$, which imply $\mathsf{nt}_A(p_{\kappa,\iota}) \circ \alpha[a_\kappa] = \alpha[a_\iota] \circ Fp_{\kappa,\iota}$ and $\mathsf{nt}_A(p_{\kappa,\iota}) \circ \mathsf{nt}_A(a_\kappa) = \mathsf{nt}_A(a_\iota)$ by the definition of the functorial extension of nt_A.

For κ a limit ordinal and $\iota < \kappa$, it is straightforward that $p_{\kappa,\iota} \circ f_\kappa = \pi_{\kappa,\iota} \circ \langle f_\iota\rangle_{\iota<\kappa} = f_\iota$.

Given that $A_\lambda = A_{\lambda+1} = A$, which implies that $p_\lambda = \mathsf{id}_A$, $a_{\lambda+1} = \mathsf{id}_A$, $\alpha[a_\lambda] = \alpha$, it is immediate that f_λ is a solution (in f) of the equation

$$\begin{array}{ccc} C & \xrightarrow{\quad\gamma\quad} & FC + A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff+\mathsf{id}_A} \\ A & \xleftarrow[{[\alpha,\mathsf{id}_A]}]{} & FA + A \end{array}$$

Indeed, $f_\lambda = p_\lambda \circ f_{\lambda+1} = f_{\lambda+1} = [\alpha[a_\lambda] \circ Ff_\lambda, a_{\lambda+1}] \circ \gamma = [\alpha \circ Ff_\lambda, \mathsf{id}_A] \circ \gamma$.

To show that it is the only solution, i.e., that, for any other solution f, we have $f = f_\lambda$, we show that $a_\kappa \circ f = f_\kappa$. We do this by induction.

$a_0 \circ f = !_A \circ f = !_C = f_0$ holds trivially.

We also have $a_{\kappa+1} \circ f = \mathsf{nt}_A(a_\kappa) \circ f = \mathsf{nt}_A(a_\kappa) \circ [\alpha \circ Ff, \mathsf{id}_A] \circ \gamma = [\mathsf{nt}_A(a_\kappa) \circ \alpha \circ Ff, \mathsf{nt}_A(a_\kappa)] \circ \gamma = [\alpha[a_\kappa] \circ F(a_\kappa \circ f), \mathsf{nt}_A(a_\kappa)] \circ \gamma = [\alpha[a_\kappa] \circ Ff_\kappa, a_{\kappa+1}] \circ \gamma = f_{\kappa+1}$, from the induction hypothesis $a_\kappa \circ f = f_\kappa$, using that f is a solution.

For κ a limit ordinal, we get $a_\kappa \circ f = \langle a_\iota\rangle_{\iota<\kappa} \circ f = \langle a_\iota \circ f\rangle_{\iota<\kappa} = \langle f_\iota\rangle_{\iota<\kappa} = f_\kappa$ from the induction hypotheses $a_\iota \circ f = f_\iota$ (for $\iota < \kappa$).

From this basis, the desired result $f = f_\lambda$ is already immediate: as $a_\lambda = \mathsf{id}_A$, it is trivial that $f = a_\lambda \circ f = f_\lambda$. □