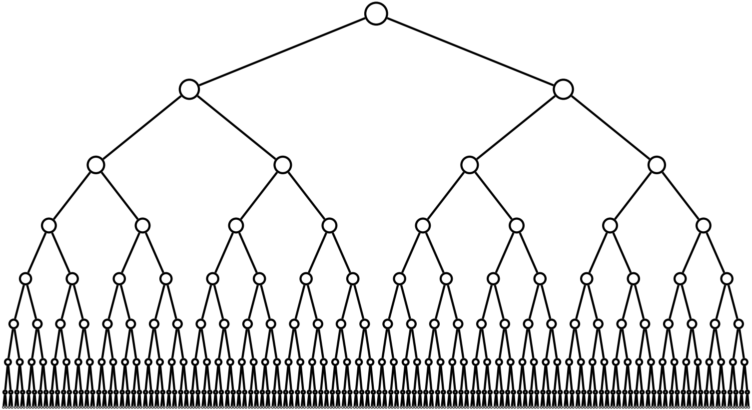
Puzzle: Colouring Infinite Trees

Consider the infinite binary tree:



Puzzle: Colouring Infinite Trees

Corecursion without Bisimulation

Venanzio Capretta

Colouring Infinite Trees

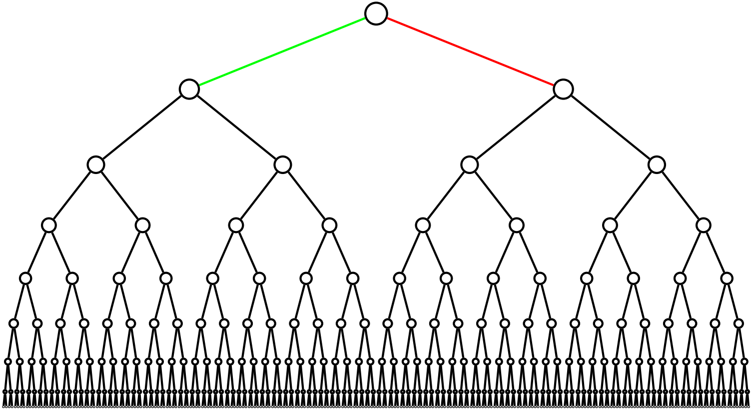
Concepts

pCRA

Bisimulation

Relation between pCRA and DA

Colour the edges green and red.



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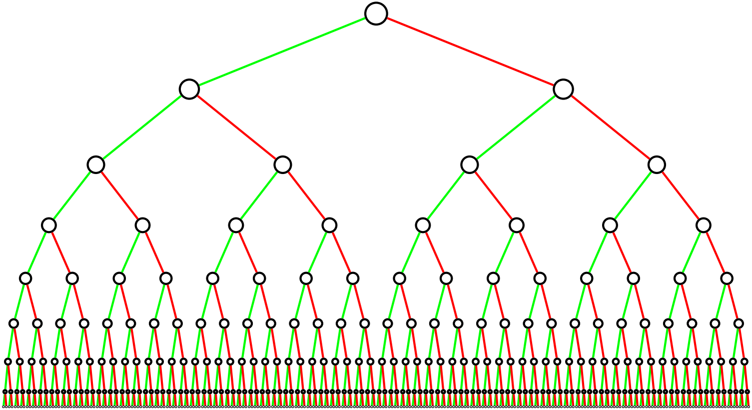

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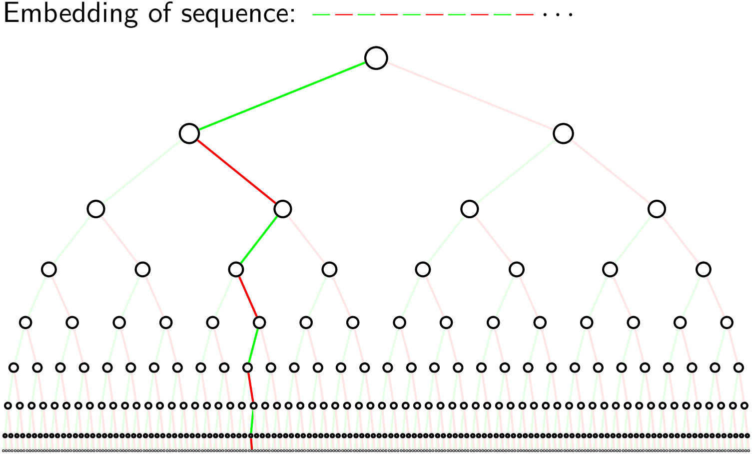
Embedding of sequence:



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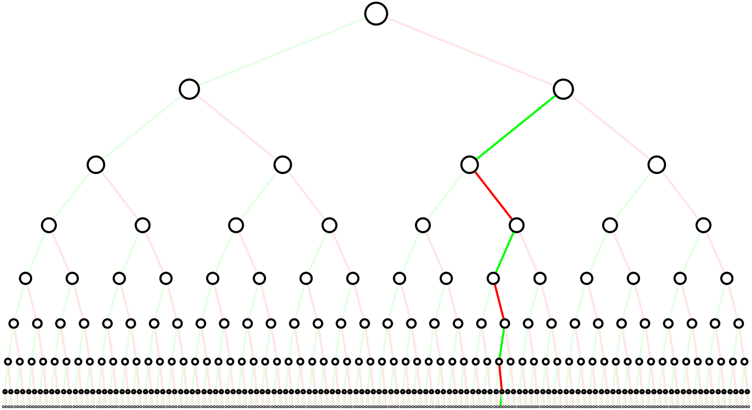

Puzzle: colouring the infinite binary tree

Problem:
Find a colouring of the tree such that:
Every infinite coloured path is embedded at most once.

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- Infinite number of colours: Trivial.

Corecursion without Bisimulation

Venanzio Capretta

Nijmegen, 18 March 2008

Work with

- Tarmo Uustalu, IoC Tallinn and
- Varmo Vene, University of Tartu

The main concepts.

In developing a theory of computation over infinite structures we need the following concepts:

The main concepts.

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- Counterexample using infinite trees.

Infinite structures as algebras

An algebra for streams: (for the functor $FX = \mathbb{N} \times X^2$)

$$\alpha : \mathbb{N} \times \mathbb{S}^2 \to \mathbb{S}$$
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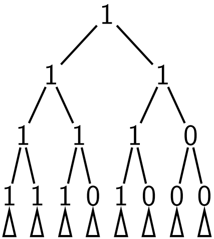

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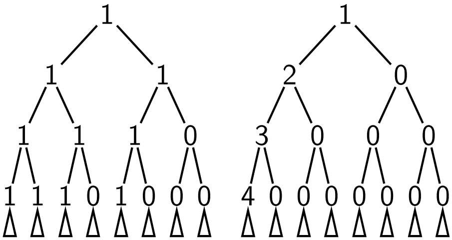

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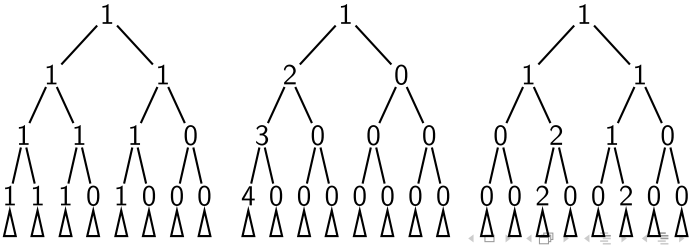

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Decomposition of f: $f = \alpha \circ Ff \circ \beta$

with:

$$\beta : \mathbb{N} \to \mathbb{N} \times \mathbb{N}^2$$
$$\beta\ x = \langle x, 2x, 2x+1 \rangle$$

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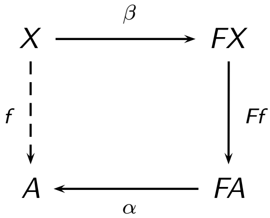
An algebra $\alpha : A \to FA$ (for a functor F) is called *corecursive (CRA)* if every recursive diagram:



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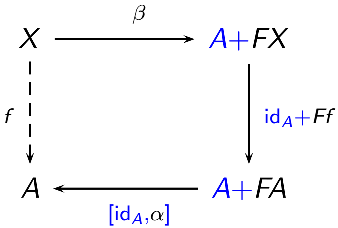

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Example of pCRA

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&g : \mathbb{B} \to \mathbb{S} \\
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\beta : \mathbb{B} \to \mathbb{S} + \mathbb{N} \times \mathbb{B}^2 \\
\beta\,\mathsf{true} = \mathsf{inl}\,(1 :: 2 :: 3 :: 4 :: \cdots) \\
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green/red algebras

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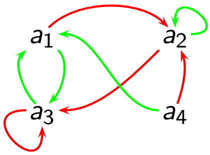

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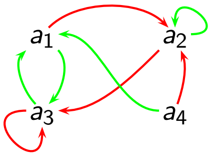

When is an algebra (parametrically) corecursive?

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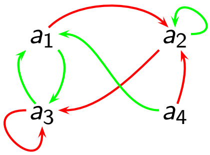

When is an algebra (parametrically) corecursive?

When Every infinite backward red/green sequence of arrows (example: $\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\cdots$) can be embedded exactly once.

Bisimulation

Informal notion of bisimulation:
Two objects are bisimilar if

- all their observable properties are the same;
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$\cong$ is the coarsest relation satisfying these properties.

The next-time operator

Definition (Next-time operator)

$(A, \alpha : FA \to A)$ algebra. Let $\equiv$ be an equivalence on A.

The next time relation $\stackrel{\mathsf{nt}}{\equiv}$ is the reflexive/transitive closure of:

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In categorical terms:

Given an epimorphism $q : A \twoheadrightarrow Q$ $(Q = A/\equiv)$;

$\mathsf{nt}(Q)$ is defined by a pushout:

$$\begin{array}{ccc} FA & \xrightarrow{\alpha} & A \\ {\scriptstyle Fq}\downarrow & & \downarrow{\scriptstyle \mathsf{nt}(q)} \\ FQ & \xrightarrow[\alpha[q]]{} & \mathsf{nt}(Q) \end{array}$$

green/red bisimulation

Example:

$(A, \mathsf{green} : A \to A, \mathsf{red} : A \to A)$ $(\alpha : A + A \to A)$

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Categorically: if $\mathrm{nt}(q)$ factorizes through q, then q is iso.

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Counterexample exploiting transitive closure.

pCRA but not DA

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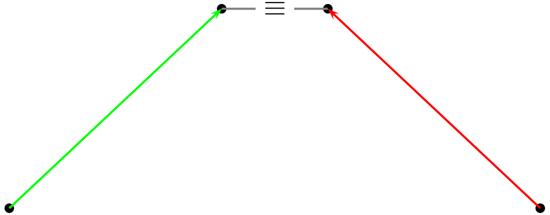

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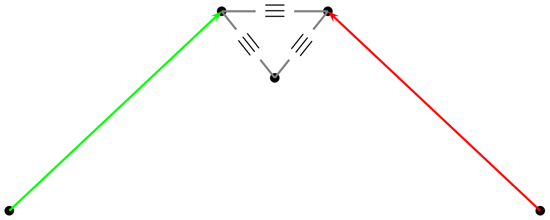

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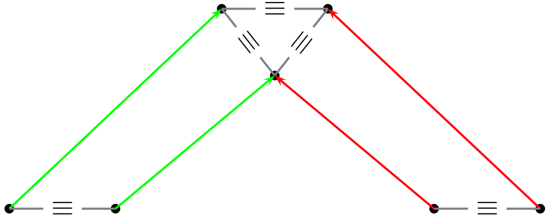

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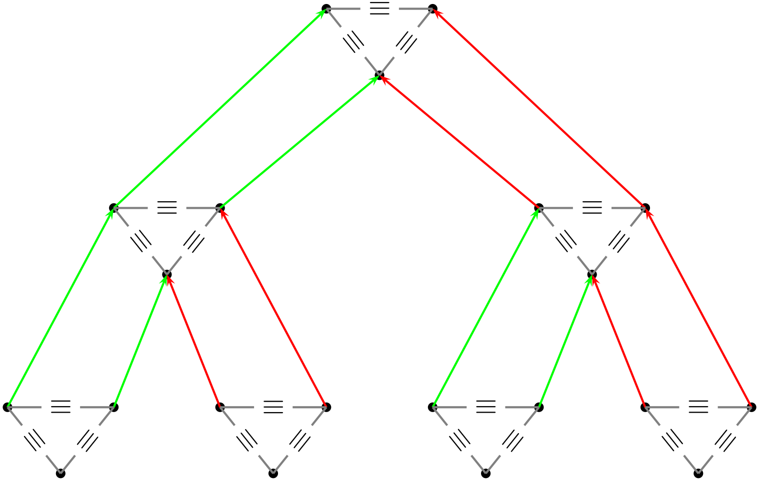

The Counterexample

Actual counterexample:

- Algebra for the functor $FX = \mathbb{N} \times X$;
- Satisfies pCRA but not DA;
- Contains an infinite tree:
  - Each node contains three distinct elements;
  - The three elements are bisimilar;
  - Inherent use of transitive closure.

Formal details:
Venanzio Capretta, Tarmo Uustalu, and Varmo Vene,
Corecursive Algebras: A Study of General Structured Corecursion