LPO Proofs in Two Educational Contexts

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Keywords: Term Rewriting Systems, Lexicographical Path Order, Tree Style Proof, Fitch Style Proof, Digital Exams.

Abstract: The purpose of this paper is twofold. First, it introduces several styles for constructing and writing down mathematical proofs for a specific technique used in theoretical Computing Science. Second, an inventory of pros and cons of these proof styles is made in two educational contexts, namely whether the proof styles help students in understanding the proofs, and whether the proof styles are practical in both written and digital exams. It turns out that there is no clear winner in both contexts, but the newly introduced so-called *shuffled Fitch style* is the most practical choice.

1 INTRODUCTION

For several years the author has been involved in teaching an introductory course on Term Rewriting Systems (TRS) and λ-calculus to second-year Computing Science students. With respect to the TRS part, basically three topics are discussed: reduction, termination, and confluence. For the termination part, two proof methods are introduced: the semantical method of polynomial monotonic interpretations (Terese, 2003), and the syntactical method of lexicographic path order, or LPO in short, (Kamin and Levy, 1980; Dershowitz, 1982; Terese, 2003).

After introducing these methods and having the students do a homework assignment about this topic, the author always asks which method is preferred by the students. And typically, they vote like 90% in favor of the polynomial monotonic interpretations. Even though they know that one of the benefits of the LPO method is that they can build the requested order *on the fly*: even if students don't have a clue about a useful order, they can simply start applying the rules and then derive such an order. At the exam, often the TRS to be studied is chosen in such a way that both techniques can be used for proving termination of the system. And again, most students opt for the polynomial monotonic interpretations. For instance, in the exam of 2020, 34 out of 42 students chose the method of polynomial monotonic interpretations. However, every now and then there is an exam where students are enforced to apply the LPO method, and this usually leads to bad results. Students seem to get lost in all the nested applications of the definition of $>_{lpo}$ as given below in Definition 1.

In order to solve this problem a few different styles of visualizations of such an LPO proof are presented in this paper. After describing some benefits and drawbacks with respect to the educational objective of letting the students understand the proofs better, it is also discussed whether these proof styles can be used easily in both written and digital exams.

Because this paper is not about comparing the two different methods for proving termination, but specifically about improving the didactics for the LPO method, we do not explain the details for the polynomial monotonic interpretations method.

2 THE LPO THEORY

The method of LPO was introduced in (Kamin and Levy, 1980). Later on, many slightly different variants have been published, for instance in (Dershowitz, 1982; Baader and Nipkow, 1998; Terese, 2003). Actually, some of these versions are considered more practical than Definition 1 below, however, this paper is about the author's course which happens to use course notes (Zantema, 2014) with the following definition:

Definition 1 ($>_{lpo}$)**.** Let $>$ be an order on the set of function symbols. Let f and g be function symbols. And let $t_1, \ldots, t_n, u, u_1, \ldots, u_m$ be terms. Then $f(t_1,\ldots,t_n) >_{lpo} u$ if and only if

1. there exists $i \in \{1,\ldots,n\}$ such that

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(a) $t_i = u$, or

(b) $t_i >_{lpo} u$

or

2. $u = g(u_1, \dots, u_m)$ and for all $i \in \{1, \dots, m\}$ it holds that $f(t_1, \dots, t_n) >_{lpo} u_i$ and either

(a) $f > g$, or

(b) $f = g$ and $(t_1, \dots, t_n) >_{lpo}^{lex} (u_1, \dots, u_m)$.

Rule 2b refers to $>_{lpo}^{lex}$, which is defined as follows:

Definition 2 ($>_{lpo}^{lex}$)**.** Let $t_1, \dots, t_n, u_1, \dots, u_m$ be terms. Then $(t_1, \dots, t_n) >_{lpo}^{lex} (u_1, \dots, u_m)$ if and only if

1. $n = m$, and
2. there exists $i \in \{1, \dots, n\}$ such that

(a) $t_i >_{lpo} u_i$ and

(b) for all $j \in \{1, \dots, i-1\}$ it holds that $t_j = u_j$.

And this is the main theorem behind the LPO method:

Theorem 3. *Let R be a TRS. If $\ell >_{\text{lpo}} r$ for all $\ell \to r$ in R, then R is terminating.*

So in order to prove that $\ell >_{lpo} r$ for some $\ell \to r$ in R, students have to apply Rules 1b, 2a and 2b repeatedly until they end up with a proof obligation for which they can apply one of the closing rules, Rules 1a, 2a and 2b.

As a running example in this paper, we will apply the technique on the two-parameter version of the so-called Ackermann function (Ackermann, 1928; Péter, 1935; Robinson, 1948):

Example 4. Consider the TRS R defined by these rules:

$$\begin{aligned} A(0,n) &\to s(n) \\ A(s(m),0) &\to A(m,s(0)) \\ A(s(m),s(n)) &\to A(m,A(s(m),n)) \end{aligned}$$

Show that R is terminating by using the lexicographic path order given that $A > s > 0$. Because of Theorem 3 we know that it suffices to show that:

1. $A(0,n) >_{lpo} s(n)$
2. $A(s(m),0) >_{lpo} A(m,s(0))$
3. $A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$

3 PROOF STYLES

We now continue by presenting proofs of termination of the TRS R from Example 4 using different styles.

3.1 Natural Language Style Proofs

In the literature, LPO proofs are usually given in the form of natural languages proofs, for instance in the classic works (Baader and Nipkow, 1998) and (Terese, 2003). Proofs in natural language can be quite diverse with respect to their formal counterparts. Because the first claim is intrinsically easy to prove, even a very informal usage of natural language will suffice. However, a proof for the second claim in an informal style, will be more difficult to follow:

> Claim $A(s(m),0) >_{lpo} A(m,s(0))$ holds because we can apply Rule 2b. This is allowed because both the left-hand side term and the right-hand side term start with the same function symbol A, in combination with the fact that $A(s(m),0) >_{lpo} m$, $A(s(m),0) >_{lpo} s(0)$, and $(s(m),0) >_{lpo}^{lex} (m,s(0))$. The first of these three facts follows directly from Rule 1a. The second fact follows from Rule 2a, because …(and so on)

Although this kind of prose is not wrong, students are advised to use a more formalized approach using explicit references when writing natural language proofs. This is what that could look like for the proof of the third claim:

1. Claim $A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$ holds because we can apply Rule 2b. Therefore, we have to prove these claims:
   - (a): $A(s(m),s(n))$ and $A(m,A(s(m),n))$ start with the same function symbol,
   - (b): $A(s(m),s(n)) >_{lpo} m$,
   - (c): $A(s(m),s(n)) >_{lpo} A(s(m),n)$, and
   - (d): $(s(m),s(n)) >_{lpo}^{lex} (m,A(s(m),n))$.
2. Claim (a) holds because both terms start with A.
3. Claim (b) holds because we can apply Rule 1b. Therefore, we have to prove a single claim:
   - (b.a): $s(m) >_{lpo} m$.
4. Claim (b.a) follows from applying Rule 1a.
5. Claim (c) holds because we can apply Rule 2b. Therefore, we have to prove three claims:
   - (c.a): $A(s(m),s(n)) >_{lpo} s(m)$,
   - (c.b): $A(s(m),s(n)) >_{lpo} n$, and
   - (c.c): $(s(m),s(n)) >_{lpo}^{lex} (s(m),n)$.
6. Claim (c.a) follows directly from Rule 1a.
7. And so on for claims (c.b), (c.c), and (d).

This proof is written in the order that these kind of proofs are typically created, the so-called *top-down*

$$\frac{}{f(t_1,\ldots,t_n) >_{lpo} t_i}\ \text{1a-}i \qquad \frac{t_i >_{lpo} u}{f(t_1,\ldots,t_n) >_{lpo} u}\ \text{1b-}i$$

$$\frac{f(t_1,\ldots,t_n) >_{lpo} u_1 \quad \ldots \quad f(t_1,\ldots,t_n) >_{lpo} u_m}{f(t_1,\ldots,t_n) >_{lpo} g(u_1,\ldots,u_m)}\ \text{2a-}f > g$$

$$\frac{f(t_1,\ldots,t_n) >_{lpo} u_1 \quad \ldots \quad f(t_1,\ldots,t_n) >_{lpo} u_m \quad (t_1,\ldots,t_n) >_{lpo}^{lex} (u_1,\ldots,u_m)}{f(t_1,\ldots,t_n) >_{lpo} f(u_1,\ldots,u_m)}\ \text{2b-}f$$

$$\frac{t_i >_{lpo} u_i}{(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n) >_{lpo}^{lex} (t_1,\ldots,t_{i-1},u_i,u_{i+1},\ldots,u_n)}\ \text{lex-}i$$

Figure 1: Derivation rules corresponding to $>_{lpo}$ and $>_{lpo}^{lex}$.

construction: starting with the main complex goal, dividing it up into simpler goals step by step. However, writing out such a proof in this order in a clear way, is more difficult, due to the nesting of rule applications, which sort of enforces to use complex references like (c.b). It turns out to be more easy to write out a proof starting with the simpler statements and building up to the final complex conclusion, the so-called *bottom-up presentation*.

1. From Rule 1a it follows that $s(m) >_{lpo} m$.
2. From claim 1 and Rule 1b it follows that $A(s(m),s(n)) >_{lpo} m$.
3. From the application of Rule 1a it follows that $A(s(m),s(n)) >_{lpo} s(m)$.
4. From Rule 1a it follows that $s(n) >_{lpo} n$.
5. From claim 4 and Rule 1b it follows that $A(s(m),s(n)) >_{lpo} n$.
6. From Rule 1a it follows that $s(n) >_{lpo} n$.
7. From claim 6 and the definition of $>_{lpo}^{lex}$ it follows that $(s(m),s(n)) >_{lpo}^{lex} (s(m),n)$.
8. From 3, 5, 7 and Rule 2b, it follows that $A(s(m),s(n)) >_{lpo} A(s(m),n)$.
9. And so on for the remaining steps...

3.2 Tree Style Proofs

Students taking this particular course, all followed a different course about logic where the theory of *natural deduction* was the main topic. In particular, the proofs in that course were presented in Gentzen tree style. Students do not really like the fact that these proofs tend to become quite wide, but they do like the fact that the structure of the proof is completely clear. So why not introduce tree style proofs for LPO as well? Note that this has been done in the past as well, for instance in (Cichon and Marion, 2000).

The definitions are quite naturally transformed into derivation rules:

Definition 5. Let $>$ be an order on the set of function symbols. Let f and g be function symbols. And let $t_1, \ldots, t_n, u, u_1, \ldots, u_m$ be terms. Then the derivation rules corresponding to Definition 1 and Definition 2 are given in Figure 1.

These rules require some explanation:

- In Rule 1a, the 'there exists $i \in \{1,\ldots,n\}$' is made explicit in the name of the rule. Maybe a more natural alternative would have been:

$$\frac{t_i = u}{f(t_1,\ldots,t_n) >_{lpo} u}\ \text{1a}$$

  However, in the proofs that would lead to rather trivial proof obligations of the form $u = u$, which would need an additional reflexivity axiom to actually close the branch. (Note that this approach is taken in (Cichon and Marion, 2000).) Therefore, we just include the index i in the name of the rule and replace the u below the line by t_i.
- In Rule 1b we do write t_i explicitly above the line, because in this case there may still be a complex proof needed above this rule.
- In Rule 2a the characterizing part is the proof obligation $f > g$. This could have been made explicit by adding it above the line, but then we would need another axiom to close this branch. And by listing it in the name of the rule, we still have a clear place in the proof where we know that we can only apply this rule, if the given order indeed implies that $f > g$. Note that if g happens to be a function symbol with no arguments, then Rule 2a introduces no new proof obligations and acts in fact like an axiom.
- In Rule 2b we do not write the proof obligation $f = g$ above the line, but enforce this already below the line by replacing the original g by f, which implies that you can only apply this rule if

Proof of $A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$:

$$\dfrac{\dfrac{\dfrac{}{s(m) >_{lpo} m}\;\text{1a-1}}{A(s(m),s(n)) >_{lpo} m}\;\text{1b-1} \qquad T_1 \qquad \dfrac{\dfrac{}{s(m) >_{lpo} m}\;\text{1a-1}}{(s(m),s(n)) >^{lex}_{lpo} (m,A(s(m),n))}\;\text{lex-1}}{A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))}\;\text{2b-}A$$

where T_1 is an abbreviation for:

$$\dfrac{\dfrac{}{A(s(m),s(n)) >_{lpo} s(m)}\;\text{1a-1} \qquad \dfrac{\dfrac{}{s(n) >_{lpo} n}\;\text{1a-1}}{A(s(m),s(n)) >_{lpo} n}\;\text{1b-2} \qquad \dfrac{\dfrac{}{s(n) >_{lpo} n}\;\text{1a-1}}{(s(m),s(n)) >^{lex}_{lpo} (s(m),n)}\;\text{lex-2}}{A(s(m),s(n)) >_{lpo} A(s(m),n)}\;\text{2b-}A$$

Figure 2: Tree style termination proof for Example 4.

the leading function symbols are the same. Just like in Rule 2a, if f is a function symbol with no arguments, this rule actually operates like an axiom.

- Also in the rule for the definition of $>^{lex}_{lpo}$ we enforce the obligation that $n = m$ by replacing the original m by n below the line. In addition, we also do not include the proof obligations $t_1 = u_1$, $\ldots$, $t_{i-1} = u_{i-1}$ above the line, but enforce these already below the line by replacing $u_1,\ldots,u_{i-1}$ explicitly by $t_1,\ldots,t_{i-1}$.

As indicated before, if we want to prove that the TRS from Example 4 is terminating, we have to prove three claims. Due to space constraints in this paper, we only provide the tree for the proof of the third claim in Figure 2. Note that we used an abbreviation T_1 for a subproof, because otherwise the proof would be too wide to fit. The students of this course are used to this kind of abbreviations.

Note that the bottom-up presentation in natural language given before, exactly coincides with the left branch and the T_1 branch!

Because the trees with their subtrees clearly resemble the proof obligations from Definition 1 and Definition 2, the conclusion is that the tree style proofs really help in clarifying the structure of the proof.

3.3 Fitch Style Proofs

So the introduced tree style proof is clear and understandable to students, mainly because they have seen proof trees before. However, it has the problem that the proofs can typically become very wide. And on paper 'wide' means trouble. But proofs that become 'tall' usually cause less trouble. So it seems reasonable to use the same solution that is common to natural deduction proofs in general: use Fitch notation.

There are several slightly different versions of this type of proofs, but in general they are all called Fitch style. See (Pelletier, 1999a) for an overview. What is the general idea? Proofs are linear, numbered lists of propositions, starting with assumptions at the top and conclusions at the bottom. For every (intermediate) conclusion, it is written explicitly which rule is applied and on which (lower) line numbers. In case a temporary assumption is made, this will be visualized by creating a new 'box' with this assumption at the top and the new goal again at the bottom of this 'box'. This 'box' defines the scope of the new assumption: it is only valid inside its own 'box'. The main difference between the variants of Fitch proofs is the way the 'boxes' are drawn. Sometimes, they are drawn like real boxes, sometimes only with long lines, sometimes only with short hooks. However, in the situation of LPO proofs, it doesn't really matter which specific visualization is used for 'boxes', because there are never temporary assumptions introduced, and hence there are never 'boxes' used in the proof!

The proof for the third claim, which can be found in Figure 3, looks pretty simple in this plain Fitch style. In fact, it is a slightly more formal version of the bottom-up presentation in natural language that we saw before. Note that we didn't optimize this proof. Because Fitch proofs can use any proposition that is both in scope and written above it, we could have optimized the proof a bit by removing duplicate propositions. For instance, in the proof in Figure 3 lines 4 and 6, and lines 1 and 9 are the same. So we could have removed lines 6 and 9 and change the references in lines 7 and 10 to 4 and 1 respectively. This optimization was not done in order to keep the relation with the tree style proofs more clear.

However, although these Fitch representations are fairly simple, the proofs do have some drawbacks.

1.	$s(m) >_{lpo} m$	1a-1
2.	$A(s(m),s(n)) >_{lpo} m$	1b-1 1
3.	$A(s(m),s(n)) >_{lpo} s(m)$	1a-1
4.	$s(n) >_{lpo} n$	1a-1
5.	$A(s(m),s(n)) >_{lpo} n$	1b-2 4
6.	$s(n) >_{lpo} n$	1a-1
7.	$(s(m),s(n)) >_{lpo}^{lex} (s(m),n)$	lex-2 6
8.	$A(s(m),s(n)) >_{lpo} A(s(m),n)$	2b-A 3, 5, 7
9.	$s(m) >_{lpo} m$	1a-1
10.	$(s(m),s(n)) >_{lpo}^{lex} (m,A(s(m),n))$	lex-1 9
11.	$A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$	2b-A 2, 8, 10

Figure 3: Linear Fitch style proof of the third claim $A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$.

These drawbacks are related to the reasons why the author typically prefers tree style proofs when *constructing* them on the blackboard during lectures, although a study on elementary logic text books (Pelletier, 1999b) showed that out of 33 books, only four of them use tree style proofs. First, note that proof trees can easily be created on the blackboard. For proofs in Fitch style this is more difficult, because you have to guess in advance how much vertical space is needed to complete the subproofs. Second, you can only fill in the reference numbers when the full proof is finished, because they typically change when creating the proof. So at the end you have to be very precise in finding the correct lines to reference, which can easily go wrong because the clear structure of the tree style proofs is lost. So for longer proofs, it can be difficult to find the proper references at the end.

This problem of lack of clear structure can easily be solved by adjusting the presentation by using 'nested hooks', for indicating the branches of the original tree. The result for the proof of the third claim is displayed in this so-called 'hooked Fitch style' in Figure 4. Now that the structure is clear again, it is much easier to check the reference numbers, because for single references they simply refer to numbers higher in the current hook, and for compound rules, they always refer to the last lines of all the hooks above on the current level.

1.	$s(m) >_{lpo} m$	1a-1
2.	$A(s(m),s(n)) >_{lpo} m$	1b-1 1
3.	$A(s(m),s(n)) >_{lpo} s(m)$	1a-1
4.	$s(n) >_{lpo} n$	1a-1
5.	$A(s(m),s(n)) >_{lpo} n$	1b-2 4
6.	$s(n) >_{lpo} n$	1a-1
7.	$(s(m),s(n)) >_{lpo}^{lex} (s(m),n)$	lex-2 6
8.	$A(s(m),s(n)) >_{lpo} A(s(m),n)$	2b-A 3, 5, 7
9.	$s(m) >_{lpo} m$	1a-1
10.	$(s(m),s(n)) >_{lpo}^{lex} (m,A(s(m),n))$	lex-1 9
11.	$A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$	2b-A 2, 8, 10

Figure 4: Fitch style proof of the third claim $A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$ with hooks for subproofs.

The third variant of a Fitch style proof that is introduced, tries to solve the problems that students have with the two variants that are already mentioned before. In the linear Fitch style, the proofs are constructed top-down, but presented bottom-up. In addition, students find it difficult to get the reference numbers right, especially without the hooks. This so-called 'top-down Fitch style' overcomes both problems by using nested hooks again but this time in a top-down presentation. See Figure 5. Unfortunately, it doesn't solve the vertical space guessing problem.

$A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$
Apply rule 2b-A; to prove:

$A(s(m),s(n)) >_{lpo} m$
Apply rule 1b-1; to prove:

$s(m) >_{lpo} m$
Apply rule 1a-1; done.

$A(s(m),s(n)) >_{lpo} A(s(m),n)$
Apply rule 2b-A; to prove:

$A(s(m),s(n)) >_{lpo} s(m)$
Apply rule 1a-1; done.

$A(s(m),s(n)) >_{lpo} n$
Apply rule 1b-2; to prove:

$s(n) >_{lpo} n$
Apply rule 1a-1; done.

$(s(m),s(n)) >_{lpo}^{lex} (s(m),n)$
Apply rule lex-2; to prove:

$s(n) >_{lpo} n$
Apply rule 1a-1; done.

$(s(m),s(n)) >_{lpo}^{lex} (m,A(s(m),n))$
Apply rule lex-1; to prove:

$s(m) >_{lpo} m$
Apply rule 1a-1; done.

Figure 5: Top-down Fitch style proof of the third claim with hooks instead of references. Because the visualization of the nesting is more important than the actual claims, we took the liberty to reduce the size a bit.

The last variant that is presented here, was brought to the author's attention by Cynthia Kop when discussing this paper. And because her method scores very well on the evaluation criteria introduced later on in Section 5, it was decided to include it as well, although it was not used by the author yet in his course. In this paper this method is referred to as the 'shuffled Fitch style'. The reason for this name will become clear after seeing an example.

So how does it work? First, one writes down the original proof obligation on the first line, without any references behind it. Then, one selects the first line that has no proof justification behind it, one applies the appropriate rule on it, and one writes the

1.	$A(s(m),s(n)) >_{lpo} A(m,A(s(m),n))$	2b-A 2, 3, 4
2.	$A(s(m),s(n)) >_{lpo} m$	1b-1 5
3.	$A(s(m),s(n)) >_{lpo} A(s(m),n)$	2b-A 6, 7, 8
4.	$(s(m),s(n)) >^{lex}_{lpo} (m,A(s(m),n))$	lex-1 5
5.	$s(m) >_{lpo} m$	1a-1
6.	$A(s(m),s(n)) >_{lpo} s(m)$	1a-1
7.	$A(s(m),s(n)) >_{lpo} n$	1b-2 9
8.	$(s(m),s(n)) >^{lex}_{lpo} (s(m),n)$	lex-2 9
9.	$s(n) >_{lpo} n$	1a-1

Figure 6: Shuffled Fitch style proof of the third claim.

new proof obligations under the existing list. The name of the applied rule (in this case 2b-A), together with the newly generated line numbers (2, 3, and 4) can be written down immediately, because they won't change anymore. And this process is repeated. So in the next step the second proof obligation is the first without a proof justification. So the appropriate rule is applied, a single new proof obligation is added, and the proof justification (1b-1 5) is written down. This process continues until all proof obligations have a proof justification. See Figure 10 for the full construction.

Once the total proof is finished, the first thing that is noticeable is that the proof is two steps shorter than the previous proofs. This is due to the fact that reusing lines 5 and 9 comes natural in this method, whereas in the previous methods it would have been possible to do that as well, but in a less natural way. For instance, in tree style it is very uncommon to reuse results from a different branch. The second noticeable thing is that it really looks like the Fitch proof in Figure 3, except for the fact that the order is no longer enforced by the structure of the proof, but by the order in which particular proof obligations are justified. When introduced above, the method stated that each time the first obligation without a justification should be taken care of, but that was an arbitrary choice for ease of explaining. The last line, or a random line would have also worked. This results in a shuffled order of a normal Fitch proof, and hence the name.

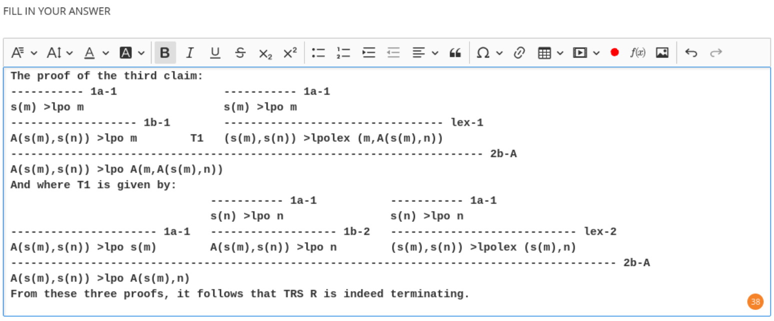
4 PROOFS IN DIGITAL EXAMS

In the previous sections, the styles presented were mainly introduced with the focus on the clarity and usability when being written down on paper. And for this it doesn't really matter whether it is written on paper as part of a homework assignment or as part of an exam. They are all usable in these situations, although the more formal notations are easier to check for actual correctness. The question, however, is whether these styles are also usable in digital exams.

A few years ago digital exams were introduced at the author's institute. One of the reasons for this was the increasing number of students, making it ever more work to grade exams. And having a system that provides a good way of at least partial automatic grading of student submissions saves a lot of time. There are several environments for organizing digital exams, for instance Inspera Assessment (Nordic Assessment Innovators, 2020), RemindoToets (Paragin, 2020), TestVision (Teelen, 2020), Cirrus Assessment (Cirrus BV, 2020), and WISEflow (UNIwise, 2020). The system currently in use at the author's institution is the Cirrus Assessment software. Therefore, the rest of this paper is about usability in Cirrus, but presumably the results will also hold for other digital examination systems.

Cirrus Assessment is cloud based: it can be used from campus, but also from home. It used to be that all exams at the author's institute were taken in on campus lecture rooms in order to have controlled circumstances. However, due to the COVID19 pandemic, many exams are nowadays taken at home as well. In this paper it is not discussed which measures are taken to control the circumstances when the students are taking the exam at home. But it is important to stress that due to this pandemic, many courses that were scheduled to have a regular written exam on campus, now had to be converted to a digital exam. So there is a need for dealing with mathematical proofs like the ones in this paper in digital assessments.

The Cirrus software allows several types of questions that allow automatic grading, like multiple-choice questions, multiple-response questions, select from a list, fill in a blank, or even matching questions. However, the system was clearly not created with mathematical exams in mind. Although for some mathematical questions it is really well possible to redesign them a bit in order to check the same learning objective as before, but now in a way that can be automatically graded, for many other questions that is just not possible. Recently, Cirrus has added a particular 'mathematical question' that can be automatically graded, based on the platform Sowiso (Sowiso BV, 2020). This platform is connected to the computer algebra system Maxima (Shelter, 1982), which allows for randomization and complex evaluation of the student's answers. However, also this type of question is not very suitable for many mathematical problems. In particular, questions where diagrams or figures should be created to answer the question are difficult to answer. Fortunately, Cirrus imple-



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FILL IN YOUR ANSWER

The proof of the third claim:
----------- 1a-1                ----------- 1a-1
s(m) >lpo m                     s(m) >lpo m
------------------- 1b-1        --------------------------------- lex-1
A(s(m),s(n)) >lpo m        T1   (s(m),s(n)) >lpolex (m,A(s(m),n))
----------------------------------------------------------------------- 2b-A
A(s(m),s(n)) >lpo A(m,A(s(m),n))
And where T1 is given by:
                              ----------- 1a-1           ----------- 1a-1
                              s(n) >lpo n                s(n) >lpo n
---------------------- 1a-1   ------------------- 1b-2   ---------------------------- lex-2
A(s(m),s(n)) >lpo s(m)        A(s(m),s(n)) >lpo n        (s(m),s(n)) >lpolex (s(m),n)
------------------------------------------------------------------------------------------- 2b-A
A(s(m),s(n)) >lpo A(s(m),n)
From these three proofs, it follows that TRS R is indeed terminating.
```

Figure 7: Tree style proof for the third claim created in Cirrus.

mented two question types that can be used for this kind of questions, namely the 'essay question' and the 'file response question'. The last one basically transforms the exam into a regular written exam, because students can write things on paper, take a picture of it, and upload it into Cirrus as an answer. This works pretty well for 'at home' exams, but it usually cannot be used for 'on campus' exams, because the controlled circumstances disallow the usage of smartphones at all and there is no alternative available for scanning student's notes. Therefore, the focus in this paper is about the usability of the 'essay question' for submitting LPO proofs.

4.1 Natural Language Style Proofs in Cirrus

When answering an 'essay question', students get a 'rich text editor' with the usual features like font selection, markup like bold, italics, superscript and subscript, and lists, both with and without numbers. It won't be a surprise that this editor is suitable for typing proofs in natural language. Basically, the only change is that it is probably wise to replace mathematical notation like $>_{lpo}$ and $>_{lpo}^{lex}$ by '`>lpo`' and '`>lpolex`' respectively, although even this can be arranged using the superscript and subscript options.

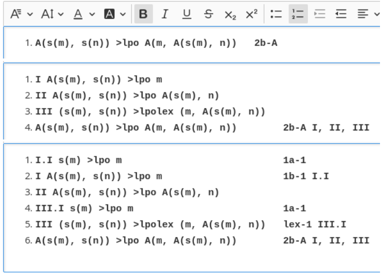
4.2 Tree Style Proofs in Cirrus

It may be a bit surprising, but within this editor it is pretty well doable to create the trees in plain ASCII. Again, replacing $>_{lpo}$ and $>_{lpo}^{lex}$ by '`>lpo`' and '`>lpolex`' respectively, makes it easier. In Figure 7 the tree for the proof of the third claim is included as it is created inside of Cirrus. The main trick in creating these trees easily within Cirrus is putting the editor into a mono-spaced font. The fact that these trees on paper are created bottom-up, is no problem, because it is easy to insert new lines above the current one. Aligning the different branches properly is also easily established by inserting the proper amount of spaces or dashes. So it is definitely doable to create tree style proofs like this in Cirrus. A clear drawback is that it takes quite some time.

4.3 Fitch Style Proofs in Cirrus

Whereas writing down Fitch proofs on paper has the problems of guessing the amount of vertical space for the subproofs and getting all the reference numbers properly, in the rich text editor this is actually no problem at all. This is because of the automatically numbered lists. Simply start with the final goal as the first item in a numbered list. In the proof of the third claim this goal is proved by applying Rule 2b, which gives three new proof obligations. Insert those above the current goal and now the proof has four lines each with their own unique label. In addition temporary references I, II, and III can be added to the last line because Rule 2b depends on those references. Those temporary labels should also be added to the proof obligations that they correspond to. As long as the proof isn't finished, the actual labels may still change and therefore these temporary labels and references are used. Once the proof is finished, all temporary references can be replaced by the actual labels corresponding to the temporary labels. And of course, all temporary labels can be removed. This process is partially shown in Figure 8. Note that this example provides also temporary labels for non-branching steps. Of course, more experienced students will notice that these are not really needed, because the final reference will always be one line above in the situation where we do not optimize for having the same proof obligations more than once.

Note that the Fitch style proof with hooks is not easy to create in the rich text editor. If normal dashes are used for showing the hooks, then the automatic numbering is broken. However, it can be done by using the 'underline' option of the editor, but that takes definitely more effort. We do not include the result in

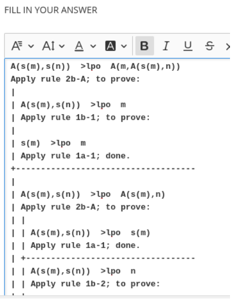


```
1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A

1. I A(s(m), s(n)) >lpo m
2. II A(s(m), s(n)) >lpo A(s(m), n)
3. III (s(m), s(n)) >lpolex (m, A(s(m), n))
4. A(s(m), s(n)) >lpo A(m, A(s(m), n))       2b-A I, II, III

1. I.I s(m) >lpo m                           1a-1
2. I A(s(m), s(n)) >lpo m                    1b-1 I.I
3. II A(s(m), s(n)) >lpo A(s(m), n)
4. III.I s(m) >lpo m                         1a-1
5. III (s(m), s(n)) >lpolex (m, A(s(m), n))  lex-1 III.I
6. A(s(m), s(n)) >lpo A(m, A(s(m), n))       2b-A I, II, III
```

Figure 8: Step by step creation of a Fitch style proof for the third claim in Cirrus. When the proof is complete all temporary roman references can be replaced by the final line numbers. The final result is an ASCII version of the proof in Figure 3.



```
FILL IN YOUR ANSWER

A(s(m),s(n))  >lpo  A(m,A(s(m),n))
Apply rule 2b-A; to prove:
|
| A(s(m),s(n))  >lpo  m
| Apply rule 1b-1; to prove:
|
| s(m)  >lpo  m
| Apply rule 1a-1; done.
+------------------------------------
|
| A(s(m),s(n))  >lpo  A(s(m),n)
| Apply rule 2b-A; to prove:
| |
| | A(s(m),s(n))  >lpo  s(m)
| | Apply rule 1a-1; done.
| +----------------------------------
| | A(s(m),s(n))  >lpo  n
| | Apply rule 1b-2; to prove:
```

Figure 9: Partial Top-down Fitch style proof for the third claim created in Cirrus.

this paper.

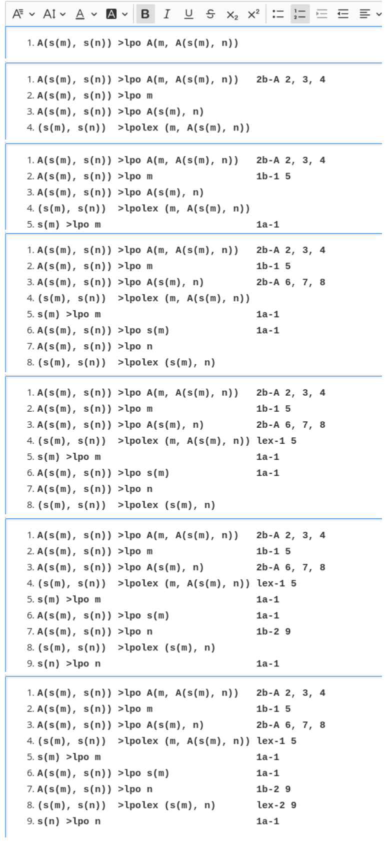
In contrast to this, the top-down Fitch style from Figure 5 turns out to be pretty easy to format in the rich text editor, because vertical and horizontal bars can be typed directly in the editor. In addition, the vertical space guessing problem is solved, because it is easy to insert more space later on. The result is presented in Figure 9.

However, the easiest method is probably the shuffled Fitch proof. Figure 10 indicates how such a proof can be created step by step. Note that we have introduced a shortcut here to immediately add the proof justification to a newly generated proof obligation if this does not require any new proof obligations, which is typically the case if we can prove a claim by Rule 1a.

5 CONCLUSION

This paper is concluded by evaluating the proposed proof styles with respect to the criteria that are important in several educational contexts:

1. Do the proofs have an understandable structure for the students?



```
1. A(s(m), s(n)) >lpo A(m, A(s(m), n))

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m
3. A(s(m), s(n)) >lpo A(s(m), n)
4. (s(m), s(n))  >lpolex (m, A(s(m), n))

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m                  1b-1 5
3. A(s(m), s(n)) >lpo A(s(m), n)
4. (s(m), s(n))  >lpolex (m, A(s(m), n))
5. s(m) >lpo m                           1a-1

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m                  1b-1 5
3. A(s(m), s(n)) >lpo A(s(m), n)         2b-A 6, 7, 8
4. (s(m), s(n))  >lpolex (m, A(s(m), n))
5. s(m) >lpo m                           1a-1
6. A(s(m), s(n)) >lpo s(m)               1a-1
7. A(s(m), s(n)) >lpo n
8. (s(m), s(n))  >lpolex (s(m), n)

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m                  1b-1 5
3. A(s(m), s(n)) >lpo A(s(m), n)         2b-A 6, 7, 8
4. (s(m), s(n))  >lpolex (m, A(s(m), n)) lex-1 5
5. s(m) >lpo m                           1a-1
6. A(s(m), s(n)) >lpo s(m)               1a-1
7. A(s(m), s(n)) >lpo n
8. (s(m), s(n))  >lpolex (s(m), n)

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m                  1b-1 5
3. A(s(m), s(n)) >lpo A(s(m), n)         2b-A 6, 7, 8
4. (s(m), s(n))  >lpolex (m, A(s(m), n)) lex-1 5
5. s(m) >lpo m                           1a-1
6. A(s(m), s(n)) >lpo s(m)               1a-1
7. A(s(m), s(n)) >lpo n                  1b-2 9
8. (s(m), s(n))  >lpolex (s(m), n)
9. s(n) >lpo n                           1a-1

1. A(s(m), s(n)) >lpo A(m, A(s(m), n))   2b-A 2, 3, 4
2. A(s(m), s(n)) >lpo m                  1b-1 5
3. A(s(m), s(n)) >lpo A(s(m), n)         2b-A 6, 7, 8
4. (s(m), s(n))  >lpolex (m, A(s(m), n)) lex-1 5
5. s(m) >lpo m                           1a-1
6. A(s(m), s(n)) >lpo s(m)               1a-1
7. A(s(m), s(n)) >lpo n                  1b-2 9
8. (s(m), s(n))  >lpolex (s(m), n)       lex-2 9
9. s(n) >lpo n                           1a-1
```

Figure 10: Step by step creation in Cirrus of a shuffled Fitch style proof for the third claim.

2. Can the proofs be *constructed* top-down in a simple way on paper?
3. Can the proofs be *presented* top-down in a simple way on paper?
4. Can the proofs be *constructed* top-down in a simple way in a rich text editor?
5. Can the proofs be *presented* top-down in a simple way in a rich text editor?

As can be seen in Table 1, there is no definitive 'best method' that scores a plus on all criteria, but the top-down Fitch style and the shuffled Fitch come close. The only problem for the first one is the necessary

Table 1: Proof styles related to criteria.

Styles \ Criteria	1	2	3	4	5
Natural language	−	±	+	±	+
Tree	+	+	−	+	−
Linear Fitch	±	±	−	+	−
Hooked Fitch	+	±	−	−	−
Top-down Fitch	+	±	+	+	+
Shuffled Fitch	±	+	+	+	+

vertical space guessing when doing the proof on paper. And the only problem for the latter one is the lack of structure in the proof. However, the shuffled Fitch style has an additional benefit that was not mentioned before: it is completely natural to write a *single* proof in this style starting with *all* three claims for the full proof, which is not the case for the top-down Fitch style! The method will work in the same easy way. So even though the shuffled Fitch style does not clearly align with the nesting structure of the proof method in general, it is very easy to present to students as an algorithm. Therefore, this is probably the best and certainly the most practical method for Computing Science students to come up with a good, easy to create, and easy to check LPO proof! If, on the other hand, the focus is on the structure of the proof, then the tree style proofs are probably the best pick, especially for this group of students that has not seen Fitch proofs before.

Note that this paper was specifically about LPO proofs and about the digital exam environment Cirrus. However, the styles can be applied for basically all mathematical proofs that rely on a series of applications of clear rules, and also in all other digital assessment environments that have a reasonable rich text editor.

6 FUTURE WORK

Because the idea for writing this paper only came up after the exam and resit of the last time the course was taught, there was no formal experiment conducted to support the conclusions in Table 1, but instead general impressions from the lectures, the input at the exam (out of the eight students using LPO, one had basically nothing, one tried a very informal natural language proof, four had a formalized natural language proof, one had a top-down Fitch style without the lines, and one had both an informal natural language proof and a top-down Fitch style without the lines because he considered his own natural language proof not clear enough), and private conversation with students. In the next round of the course, also the shuffled Fitch method will be introduced and students will probably specifically be asked to use several LPO proof visualizations in the homework and at the exam.

In addition, the question came up to give formal semantics for the shuffled Fitch method and prove that the method is sound and complete for this specific case where no normal 'boxes' are needed by lack of real assumptions. It is also interesting to check whether this shuffled Fitch method also works for proofs that do use local assumptions. Using the current informal semantics presented in Section 3.3, this doesn't seem likely because claims inside 'boxes' cannot typically be used outside these 'boxes'.

ACKNOWLEDGEMENTS

The author thanks Cynthia Kop for comments on a preliminary version and in particular for showing him the so-called 'shuffled Fitch style' proof, which actually is the most innovative and promising method discussed in this paper.

And the author thanks his wife for encouraging him to finally write down some of his ideas forthcoming from his teaching in this paper.

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