Pearl's Algorithm further Explained

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1 Fusion lemma

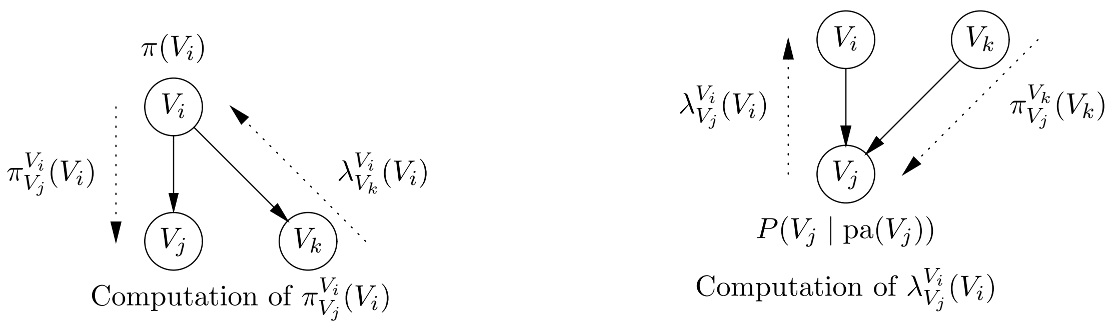

Figure 1: Pearl's message passing.

Probabilistic inference using Pearl's algorithm is done by applying the following formula to every vertex in the graph:

$P^*(V_i) = P(V_i \mid e) = \alpha \cdot \pi(V_i) \cdot \lambda(V_i)$

where:

- e: evidence
- α: normalisation constant
- $\pi(V_i) \triangleq P(V_i \mid e^+_{V_i})$: *compound* causal parameter
- $\lambda(V_i) \triangleq P(e^-_{V_i} \mid V_i)$: *compound* diagnostic parameter

with:

$$\pi(V_i) = \sum_{\mathrm{pa}(V_i)} P(V_i \mid \mathrm{pa}(V_i)) \cdot \prod_{j=1}^{m} \pi^{V_j}_{V_i}(V_j)$$

$$\lambda(V_i) = \prod_{j=1}^{m} \lambda^{V_i}_{V_j}(V_i) \quad \text{if } V_i \notin E$$

$$\pi^{V_i}_{V_j}(V_i) = \alpha \cdot \pi(V_i) \cdot \prod_{k \neq j} \lambda^{V_i}_{V_k}(V_i) \quad \text{if } V_i \notin E$$

$$\lambda^{V_i}_{V_j}(V_i) = \beta \cdot \sum_{V_j} \lambda(V_j) \cdot \sum_{V_k \in \mathrm{pa}(V_j), k \neq i} P(V_j \mid \mathrm{pa}(V_j)) \prod_{k \neq i} \pi^{V_k}_{V_j}(V_k)$$

Note that $V_i \in \mathrm{pa}(V_j)$, so when V_i is the only parent of V_j, the result of the last equation is simply $P(V_j \mid V_i)$ (without summation). See Figure 1 for an illustration. In the following section an example is discussed to show how the method works.

2 Detailed example

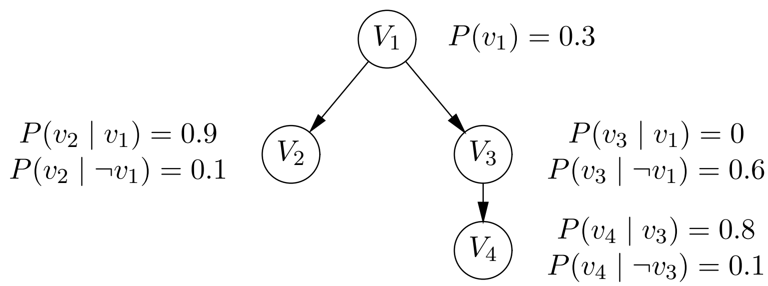

Figure 2: Example Bayesian network with evidence $V_3 = \mathit{true}$.

Consider the network shown in Figure 2 for which we study probabilistic inference using Pearl's algorithm.

$$\pi(v_1) = P(v_1) = 0.3$$

$$\pi(\neg v_1) = P(\neg v_1) = 0.7$$

$$\pi^{V_1}_{V_2}(v_1) = \pi(v_1) \cdot \lambda^{V_1}_{V_3}(v_1) = 0.3 \cdot 0 = 0$$

$$\pi^{V_1}_{V_2}(\neg v_1) = \pi(\neg v_1)\lambda^{V_1}_{V_3}(\neg v_1) = 0.7 \cdot 0.6 = 0.42$$

$$\pi^{V_1}_{V_3}(v_1) = \pi(v_1)\lambda^{V_1}_{V_2}(v_1) = 0.3$$

$$\pi^{V_1}_{V_3}(\neg v_1) = \pi(\neg v_1)\lambda^{V_1}_{V_2}(\neg v_1) = 0.7$$

$$\pi(v_2) = \sum_{V_1} P(v_2 \mid V_1)\pi^{V_1}_{V_2}(V_1) = 0.1 \cdot 0.42 = 0.042$$

$$\pi(\neg v_2) = \sum_{V_1} P(\neg v_2 \mid V_1)\pi^{V_1}_{V_2}(V_1) = 0.9 \cdot 0.42 = 0.378$$

$$\pi(v_3) = \sum_{V_1} P(v_3 \mid v_1)\lambda^{V_1}_{V_3}(v_1) = 0.6 \cdot 0.3 = 0.18$$

$$\pi(\neg v_3) = \sum_{V_1} P(\neg v_3 \mid v_1)\lambda^{V_1}_{V_3}(v_1) = 0.4 \cdot 0.3 = 0.12$$

$$\lambda(v_1) = \lambda^{V_1}_{V_2}(v_1) \cdot \lambda^{V_1}_{V_3}(v_1) = 0$$

$$\lambda(\neg v_1) = \lambda^{V_1}_{V_2}(\neg v_1) \cdot \lambda^{V_1}_{V_3}(\neg v_1) = 0.6$$

because no evidence on V_2 and evidence $V_3 = \mathit{true}$

$\lambda(v_2) = 1$ no evidence below V_2

$\lambda(\neg v_2) = 1$ no evidence below V_2

$$\lambda^{V_1}_{V_2}(v_1) = \sum_{V_2} \lambda(V_2)P(V_2 \mid v_1) = 1$$

$$\lambda^{V_1}_{V_2}(\neg v_1) = \sum_{V_2} \lambda(V_2)P(V_2 \mid \neg v_1) = 1$$

$\lambda(v_3) = 1$ evidence

$\lambda(\neg v_3) = 0$ evidence

$$\lambda^{V_1}_{V_3}(v_1) = \sum_{V_3} \lambda(V_3)P(V_3 \mid v_1)$$

$$= 1 \cdot P(v_3 \mid v_1) = 0$$

$$\lambda^{V_1}_{V_3}(\neg v_1) = \sum_{V_3} \lambda(V_3)P(V_3 \mid \neg v_1)$$

$$= 1 \cdot P(v_3 \mid \neg v_1) = 0.6$$

$$\pi_{V_4}^{V_3}(v_3) = 1 \text{ evidence} \qquad \lambda(v_4) = 1 \text{ no evidence}$$

$$\pi_{V_4}^{V_3}(\neg v_3) = 0 \text{ evidence} \qquad \lambda(\neg v_4) = 1 \text{ no evidence}$$

$$\pi(v_4) = \sum_{V_3} P(v_4 \mid V_3)\pi_{V_4}^{V_3}(V_3) = 0.8 \qquad \lambda_{V_4}^{V_3}(v_3) = \sum_{V_4} \lambda(V_4)P(V_4 \mid v_3) = 1$$

$$\pi(\neg v_4) = \sum_{V_3} P(\neg v_4 \mid V_3)\pi_{V_4}^{V_3}(V_3) = 0.2 \qquad \lambda_{V_4}^{V_3}(\neg v_3) = \sum_{V_4} \lambda(V_4)P(V_4 \mid \neg v_3) = 1$$

Application of the fusion lemma:

$$P^*(v_1) = \alpha \cdot \pi(v_1) \cdot \lambda(v_1) = \alpha \cdot 0.3 \cdot 0 = 0 \qquad P^*(\neg v_1) = \alpha \cdot \pi(\neg v_1) \cdot \lambda(\neg v_1) = \alpha \cdot 0.7 \cdot 0.6$$
$$= \alpha \cdot 0.42 = 1$$

$$P^*(v_2) = \alpha \cdot \pi(v_2) \cdot \lambda(v_2) = \alpha \cdot 0.042 \cdot 1 \qquad P^*(\neg v_2) = \alpha \cdot \pi(\neg v_2) \cdot \lambda(\neg v_2) = \alpha \cdot 0.378 \cdot 1$$
$$\Rightarrow \alpha = \frac{1}{0.42}$$
$$\Rightarrow P^*(v_2) = 0.1, P^*(\neg v_2) = 0.9$$

$$P^*(v_3) = \alpha \cdot \pi(v_3) \cdot \lambda(v_3) = \alpha \cdot 0.18 \cdot 1 \qquad P^*(\neg v_3) = \alpha \cdot \pi(\neg v_3) \cdot \lambda(\neg v_3) = \alpha \cdot 0.12 \cdot 0$$
$$\Rightarrow P^*(v_3) = 1, P^*(\neg v_3) = 0$$

$$P^*(v_4) = \alpha \cdot \pi(v_4) \cdot \lambda(v_4) = \alpha \cdot 0.8 \cdot 1 = 0.8 \qquad P^*(\neg v_4) = \alpha \cdot \pi(\neg v_4) \cdot \lambda(\neg v_4)$$
$$= \alpha \cdot 0.2 \cdot 1 = 0.2$$