Bayesian Networks 2015–2016

Tutorial II – Conditional Independence Models

Peter Lucas
LIACS, Leiden University

Introduction

The exercises below concern the independence relation $\perp\!\!\!\perp$ and graph representation of the independence relation. In contrast to the associated slides of the lecture on Markov Independence, we do not make a distinction between names of vertices in a graph, e.g., 1 and 2, and their associated random variable in a probability distribution, in this case X_1 and X_2. We will use variable names all the way through. Note that independence relations can be defined on any set of objects (vertex names of a graph, numbers, symbols), not on random variables only.

Exercises

Exercise 1

Let V be a set of random variables. Let P be a joint probability distribution of V and let $\perp\!\!\!\perp_P$ be its independence relation. Show that $\perp\!\!\!\perp_P$ satisfies the properties

a. (P.1 Symmetry) $X \perp\!\!\!\perp_P Y \mid Z \Rightarrow Y \perp\!\!\!\perp_P X \mid Z$

b. (P.2 Decomposition) $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \wedge X \perp\!\!\!\perp_P W \mid Z$

c. (P.3 Weak union) $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$

d. (P.4 Contraction) $X \perp\!\!\!\perp_P Y \mid W \wedge X \perp\!\!\!\perp_P Z \mid W \cup Y \Rightarrow X \perp\!\!\!\perp_P Y \cup Z \mid W$ (note the difference with the slides)

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.

You need to prove these properties by translating them to statements concerning the probability distribution P. For example $X \perp\!\!\!\perp_P Y \mid Z$ is first translated into $P(X \mid Y, Z) = P(X \mid Z)$.

c. We show that the independence relation $\perp\!\!\!\perp_P$ satisfies the property

$$X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$$

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.

We assume that $X \perp\!\!\!\perp_P Y \cup W \mid Z$. From this observation, we have

$$P(X \mid Z \wedge Y \wedge W) = P(X \mid Z)$$

From our assumption $X \perp\!\!\!\perp_P Y \cup W \mid Z$, we further have $X \perp\!\!\!\perp_P W \mid Z$ by the second property stated in the exercise. By definition, we therefore have that

$$P(X \mid Z \wedge W) = P(X \mid Z)$$

Now consider the conditional probability $P(X \mid Z \wedge W \wedge Y)$. We find that

$$\begin{aligned} P(X \mid Z \wedge W \wedge Y) &= P(X \mid Z) \\ &= P(X \mid Z \wedge W) \end{aligned}$$

From $P(X \mid Z \wedge W \wedge Y) = P(X \mid Z \wedge W)$, we have by definition that $X \perp\!\!\!\perp_P Y \mid Z \cup W$. We conclude that $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$.

Exercise 2

Let V be a set of random variables and let $\perp\!\!\!\perp$ be a semi-graphoid independence relation on V. Show that

$$X \perp\!\!\!\perp Y \cup W \mid Z \wedge Y \perp\!\!\!\perp W \mid Z \Rightarrow X \cup W \perp\!\!\!\perp Y \mid Z$$

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.

We begin our proof by observing that, since $\perp\!\!\!\perp$ is a semi-graphoid independence relation, it obeys the first four axioms of the independence relation $\perp\!\!\!\perp$. Now, we assume that $X \perp\!\!\!\perp Y \cup W \mid Z$ and $Y \perp\!\!\!\perp W \mid Z$. We have that

$$\begin{aligned} X \perp\!\!\!\perp Y \cup W \mid Z \quad &\Rightarrow X \perp\!\!\!\perp Y \mid Z \cup W \\ &\Rightarrow Y \perp\!\!\!\perp X \mid Z \cup W \end{aligned}$$

by the weak union and symmetry axioms; in conjunction with our assumption $Y \perp\!\!\!\perp W \mid Z$, we find

$$\begin{aligned} Y \perp\!\!\!\perp X \mid Z \cup W \wedge Y \perp\!\!\!\perp W \mid Z \quad &\Rightarrow Y \perp\!\!\!\perp W \cup X \mid Z \Rightarrow \\ &\Rightarrow X \cup W \perp\!\!\!\perp Y \mid Z \end{aligned}$$

by the contraction and symmetry axioms.

Exercise 3

Let V be a set of random variables and let $\perp\!\!\!\perp$ be a semi-graphoid independence relation on V. Show that

$$X \perp\!\!\!\perp U \cup W \mid Y \cup Z \wedge Y \perp\!\!\!\perp X \mid Z \cup U \Rightarrow X \perp\!\!\!\perp Y \cup W \mid Z \cup U$$

for all mutually disjoint sets of variables $X, Y, Z, U, W \subseteq V$.

Exercise 4

Let $V = \{X_1, X_2, X_3, X_4\}$ be a set of random variables. Furthermore, let $\perp\!\!\!\perp$ be a (semi-graphoid) independence relation on V, containing, amongst others, the following elements:

$$\begin{array}{cc} \{X_1\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2\} \mid \{X_1\} \\ \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_3\} \mid \{X_1\} \\ \{X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2, X_3\} \mid \{X_1\} \\ \{X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing & \{X_1\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \end{array}$$

$$
\begin{array}{rl}
\{X_4\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \\
\{X_4\} \perp\!\!\!\perp \{X_3\} \mid \varnothing & \{X_1, X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \\
\{X_1, X_2\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_1\} \mid \{X_2\} \\
\{X_1, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_3\} \mid \{X_2\} \\
\{X_4\} \perp\!\!\!\perp \{X_1, X_3\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_3\} \\
\{X_4\} \perp\!\!\!\perp \{X_1, X_3\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_3\} \\
\{X_1, X_2, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_1\} \perp\!\!\!\perp \{X_2\} \mid \{X_4\} \\
\{X_1\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_1, X_2\} \\
\{X_1, X_4\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_1, X_3\} \\
\{X_2, X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2\} \mid \{X_1, X_3\}
\end{array}
$$

Show that each statement $X \perp\!\!\!\perp Y \mid Z$, $X, Y, Z \subseteq V$, of the independence relation $\perp\!\!\!\perp$ can be derived from the statements $\{X_1, X_2, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing$ and $\{X_1\} \perp\!\!\!\perp \{X_2\} \mid \varnothing$, by the four independence axioms.

Exercise 5

Let $V = \{X_1, X_2, X_3, X_4\}$ be a set of random variables. Let $\perp\!\!\!\perp$ be the independence relation on V that is defined by the statements $\{X_1\} \perp\!\!\!\perp \{X_4\} \mid \{X_2, X_3\}$ and $\{X_2\} \perp\!\!\!\perp \{X_3\} \mid \{X_1, X_4\}$.

a. Give all undirected D-maps for the independence relation $\perp\!\!\!\perp$;

b. Give all undirected I-maps for the independence relation $\perp\!\!\!\perp$.

Exercise 6

Show that for any undirected graph $G = (V(G), E(G))$ the following property holds: for any vertex $X_i \in V(G)$ and any vertex $X_j \in V(G) \setminus (\{X_i\} \cup \nu_G(X_i))$, we have that $\{X_i\} \perp\!\!\!\perp_G \{X_j\} \mid \nu_G(X_i)$, where $\nu_G(X)$ is the set of neighbours in the graph G of X.

Exercise 7

Let G be the following acyclic digraph:
Examine for each of the following statements whether or not it holds in G:

a. $\{X_1\} \perp\!\!\!\perp_G^d \{X_6\} \mid \{X_2, X_3\}$;

b. $\{X_2\} \perp\!\!\!\perp_G^d \{X_3\} \mid \varnothing$;

c. $\{X_2\} \perp\!\!\!\perp_G^d \{X_3\} \mid \{X_1\}$;

d. $\{X_4\} \perp\!\!\!\perp_G^d \{X_3\} \mid \{X_1\}$;

e. $\{X_2\} \perp\!\!\!\perp_G^d \{X_6\} \mid \{X_3, X_4\}$;

f. $\{X_3\} \perp\!\!\!\perp_G^d \{X_1\} \mid \varnothing$.

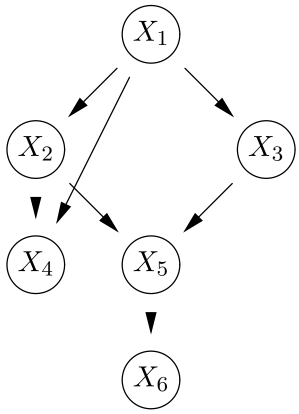

Figure 1: Bayesian network.

c. The property $\{X_2\} \perp\!\!\!\perp_G^d \{X_3\} \mid \{X_1\}$ holds in the digraph G since all chains in G from X_2 to X_3 are blocked by the set of vertices $\{X_1\}$. For example, the chain X_2, X_1, X_3 from X_2 to X_3 is blocked by $\{X_1\}$ since $X_1 \in \{X_1\}$; the chain X_2, X_5, X_3 from X_2 to X_3 is blocked by $\{X_1\}$ since $\{X_5, X_6\} \cap \{X_1\} = \varnothing$.

e. The property $\{X_2\} \perp\!\!\!\perp_G^d \{X_6\} \mid \{X_3, X_4\}$ does *not* hold in the digraph G since not every chain in G from X_2 to X_6 is blocked by the set of vertices $\{X_3, X_4\}$. For example, the chain X_2, X_5, X_6 from X_2 to X_6 is not blocked by $\{X_3, X_4\}$.

Exercise 8

Let $V = \{X_1, X_2, X_3, X_4\}$ be a set of random variables. Let $\perp\!\!\!\perp$ be the independence relation on V that is defined by the statements $\{X_1\} \perp\!\!\!\perp \{X_2\} \mid \varnothing$ and $\{X_1, X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_3\}$.

a. Give some directed D-maps for the independence relation $\perp\!\!\!\perp$;

b. Give some directed I-maps for the independence relation $\perp\!\!\!\perp$.

Exercise 9

Show that for every independence relation there exists a directed D-map and a directed I-map.

Exercise 10

Given an example of an independence relation that has more than one directed P-map.

Exercise 11

Let $V = \{X_1, X_2, X_3, X_4\}$ be a set of random variables. Let $\perp\!\!\!\perp$ be the independence relation on V that is defined by the statements $\{X_1\} \perp\!\!\!\perp \{X_4\} \mid \{X_2, X_3\}$ and $\{X_2\} \perp\!\!\!\perp \{X_3\} \mid \{X_1, X_4\}$. Give some minimal directed I-maps for the relation $\perp\!\!\!\perp$.

Exercise 12

Show that for any acyclic directed graph $G = (V(G), A(G))$ the following property holds: for any vertex $X_i \in V(G)$ and any vertex $X_j \in V(G) \setminus (\sigma^*_G(X_i) \cup \pi_G(X_i))$, we have that $\{X_i\} \perp\!\!\!\perp_G \{X_j\} \mid \pi_G(X_i)$, where $\pi(X)$ is the set of parents of vertex X and $\sigma^*_G(X)$ is the set of successors of X, including X.

Exercise 13

Let V be a set of random variables. Let $\perp\!\!\!\perp$ be an independence relation on V and let G be a directed I-map for $\perp\!\!\!\perp$. Now, let H be the underlying graph of G. Is H an undirected I-map for $\perp\!\!\!\perp$?

Exercise 14

a. Give an example of an independence relation that has both an undirected P-map and a directed P-map.

b. Give an example of an independence relation that has an undirected P-map but no directed P-map.

c. Give an example of an independence relation that has a directed P-map but no undirected P-map.

d. Give an example of an independence relation that has no undirected P-map nor a directed P-map.

Exercise 15

Consider the Bayesian network shown in Figure 1. We study in this exercise the procedure of moralisation (have a look at the slides of Lectures 3-4).

a. Determine whether or not $\{X_2\} \perp\!\!\!\perp_G \{X_3\} \mid \{X_1\}$ using moralisation.

b. Answer the same question for $\{X_2\} \perp\!\!\!\perp_G \{X_3\} \mid \{X_1, X_6\}$