Knowledge Representation and Reasoning

Logic and Resolution - Example and Exercises

Please read the relevant sections in the chapter 'Logic and Resolution' that is available on the web (http://www.cs.ru.nl/~peterl/teaching/KeR/logicintro.pdf) or blackboard to refresh your memory first.

A note on terminology. A logical formula is also called a *sentence*. The formulas or sentences that together are used to define or describe some concepts are often called *axioms*.

1. Semantics of logic

Exercise 1.1

(i) Use truth tables to prove that

$$P \to (Q \to (P \land Q))$$

is a tautology.

(ii) One of the equivalence laws of first-order predicate logic is the following:

$$\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$$

However, the following equivalence is *not* valid:

$$\forall x(P(x) \lor Q(x)) \equiv \forall x P(x) \lor \forall x Q(x)$$

Prove this by means of a counter example.

(iii) Consider the following formula in first-order logic:

$$F = \forall x \exists y (P(f(x), y) \lor \neg Q(x))$$

Suppose that the following structure S is given:

$$S = \langle D = \{a, b\}, \{g : D \to D\}, \{R : D^2 \to \{\mathit{true}, \mathit{false}\}, S : D \to \{\mathit{true}, \mathit{false}\}\}\rangle$$

where D is the domain of interpretation, and with the function g defines as follows: $g(a) = a$, and $g(b) = b$; furthermore, the predicates R and S are defined as: $R(a, a) = true$, $R(a, b) = false$, $R(b, b) = true$, $R(b, a) = false$, $S(a) = true$, and $S(b) = false$. The function g is linked to the function *symbol* f, and the predicates R and S are linked to predicate *symbols* P and Q, respectively. Determine the truth value of formula F for the given structure S.

2. Clausal form and resolution

Example
Consider the following sentences:

1. All hounds howl at night.

2. Anyone who has any cats will not have any mice.

3. Light sleepers do not have anything which howls at night.
4. John has either a cat or a hound.
5. (Conclusion) If John is a light sleeper, then John does not have any mice.

The conclusion can be proved using Resolution as shown below. The first step is to write each sentence as a well-formed formula (wff) in first-order predicate calculus. The formulae written for the above axioms are shown below, using predicate LS(x) to represent the property 'light sleeper'.

1. $\forall x(\text{Hound}(x) \rightarrow \text{Howl}(x))$
2. $\forall x \forall y(\text{Have}(x, y) \wedge \text{Cat}(y) \rightarrow \neg \exists z(\text{Have}(x, z) \wedge \text{Mouse}(z)))$
3. $\forall x(\text{LS}(x) \rightarrow \neg \exists y(\text{Have}(x, y) \wedge \text{Howl}(y)))$
4. $\exists x(\text{Have}(\text{John}, x) \wedge (\text{Cat}(x) \vee \text{Hound}(x)))$
5. $\text{LS}(\text{John}) \rightarrow \neg \exists z(\text{Have}(\text{John}, z) \wedge \text{Mouse}(z))$

The next step is to transform each wff into Prenex Normal Form, Skolemize, and rewrite as clauses in conjunctive normal form (CNF). Below we show these transformations for each first-order formula.

1. $\forall x(\text{Hound}(x) \rightarrow \text{Howl}(x))$
   $\neg \text{Hound}(x) \vee \text{Howl}(x)$
2. $\forall x \forall y(\text{Have}(x, y) \wedge \text{Cat}(y) \rightarrow \neg \exists z(\text{Have}(x, z) \wedge \text{Mouse}(z)))$
   $\forall x \forall y(\text{Have}(x, y) \wedge \text{Cat}(y) \rightarrow \forall z \neg(\text{Have}(x, z) \wedge \text{Mouse}(z)))$
   $\forall x \forall y \forall z(\neg(\text{Have}(x, y) \wedge \text{Cat}(y)) \vee \neg(\text{Have}(x, z) \wedge \text{Mouse}(z)))$
   $\neg \text{Have}(x, y) \vee \neg \text{Cat}(y) \vee \neg \text{Have}(x, z) \vee \neg \text{Mouse}(z)$
3. $\forall x(\text{LS}(x) \rightarrow \neg \exists y(\text{Have}(x, y) \wedge \text{Howl}(y)))$
   $\forall x(\text{LS}(x) \rightarrow \forall y \neg(\text{Have}(x, y) \wedge \text{Howl}(y)))$
   $\forall x \forall y(\text{LS}(x) \rightarrow \neg \text{Have}(x, y) \vee \neg \text{Howl}(y))$
   $\forall x \forall y(\neg \text{LS}(x) \vee \neg \text{Have}(x, y) \vee \neg \text{Howl}(y))$
   $\neg \text{LS}(x) \vee \neg \text{Have}(x, y) \vee \neg \text{Howl}(y)$
4. $\exists x(\text{Have}(\text{John}, x) \wedge (\text{Cat}(x) \vee \text{Hound}(x)))$
   $\text{Have}(\text{John}, a) \wedge (\text{Cat}(a) \vee \text{Hound}(a))$
5. $\neg[\text{LS}(\text{John}) \rightarrow \neg \exists z(\text{Have}(\text{John}, z) \wedge \text{Mouse}(z))]$ (negated conclusion)
   $\neg[\neg \text{LS}(\text{John}) \vee \neg \exists z(\text{Have}(\text{John}, z) \wedge \text{Mouse}(z))]$
   $\text{LS}(\text{John}) \wedge \exists z(\text{Have}(\text{John}, z) \wedge \text{Mouse}(z)))$
   $\text{LS}(\text{John}) \wedge \text{Have}(\text{John}, b) \wedge \text{Mouse}(b)$

The set of CNF clauses for this problem is thus as follows:

1. $\neg \text{Hound}(x) \vee \text{Howl}(x)$
2. $\neg \text{Have}(x, y) \vee \neg \text{Cat}(y) \vee \neg \text{Have}(x, z) \vee \neg \text{Mouse}(z)$
3. $\neg \text{LS}(x) \vee \neg \text{Have}(x, y) \vee \neg \text{Howl}(y)$
4. (a) $\text{Have}(\text{John}, a)$
   (b) $\text{Cat}(a) \vee \text{Hound}(a)$
5. (a) $\text{LS}(\text{John})$
   (b) $\text{Have}(\text{John}, b)$

(c) Mouse(b)

Now we proceed to prove the conclusion by resolution using the above clauses. Each result clause is numbered; the numbers of its parent clauses are shown at the righ-hand side.

6. Cat(a) $\vee$ Howl(a) [1, 4(b)]
7. $\neg$Have(x, y) $\vee$ $\neg$Cat(y) $\vee$ $\neg$Have(x, b) [2, 5(c)]
8. $\neg$Have(John, y) $\vee$ $\neg$Cat(y) [7, 5(b)]
9. $\neg$Have(John, a) $\vee$ Howl(a) [6, 8]
10. Howl(a) [4(a), 9]
11. $\neg$LS(x) $\vee$ $\neg$Have(x, a) [3, 10]
12. $\neg$LS(John) [4(a), 11]
13. $\square$ [5(a), 12]

Exercise 2.1
For each of the following sets of expressions, determine whether or not it is unifiable. If a given set if unifiable, then determine a most general unifier:

(i) $\{P(x, x), P(c, y)\}$

A unifier for those two expressions is a substitution σ such that $P(x, x)\sigma = P(c, y)\sigma$. We find the subexpressions in which they differ from left to right. First, c and x differ, which can be solved by replacing the variable (x) by the constant (c), which gives the substitution $\sigma_1 = \{c/x\}$. Now we have $P(x, x)\sigma_1 = P(c, c) \neq P(c, y) = P(c, y)\sigma_1$. Note that the first subexpression is the same. The remaining subexpression that differs is c and y, so we extend the substition to $\sigma_2 = \{c/x, c/y\}$. Then, $P(x, x)\sigma_2 = P(c, c) = P(c, y)\sigma_2$.

(ii) $\{P(a, x, f(x)), P(x, y, x)\}$

(iii) $\{P(x, f(y), y), P(w, z, g(a, b))\}$

(iv) $\{P(x, z, y), P(x, z, x), P(a, x, x)\}$

(v) $\{P(z, f(x), b), P(x, f(a), b), P(g(x), f(a), y)\}$

Exercise 2.2
Convert the following first-order sentences into sets of clauses:

(i) $\forall x \forall y (isgrandparent(x, y) \rightarrow \exists z (parent(x, z) \wedge parent(z, y)))$

See page 28 for a complete list with steps and another example. In this case it is as follows:

$$\begin{aligned}
& \forall x \forall y (isgrandparent(x,y) \rightarrow \exists z (parent(x,z) \wedge parent(z,y))) \\
& \text{(step 1: remove implication)} \\
\Rightarrow\ & \forall x \forall y (\neg isgrandparent(x,y) \vee \exists z (parent(x,z) \wedge parent(z,y))) \\
& \text{(step 4: remove the existential quantor (Skolemize))} \\
\Rightarrow\ & \forall x \forall y (\neg isgrandparent(x,y) \vee (parent(x,f(x,y)) \wedge parent(f(x,y),y))) \\
& \text{(step 6: convert to conjunctive normal form)} \\
\Rightarrow\ & \forall x \forall y (\neg isgrandparent(x,y) \vee parent(x,f(x,y))) \\
& \wedge (\neg isgrandparent(x,y) \vee parent(f(x,y),y)) \\
& \text{(step 7: disregard the prenex)} \\
\Rightarrow\ & (\neg isgrandparent(x,y) \vee parent(x,f(x,y))) \\
& \wedge (\neg isgrandparent(x,y) \vee parent(f(x,y),y)) \\
& \text{(step 8: write as clauses, e.g. using the implication form)} \\
\Rightarrow\ & \{parent(x,f(x,y)) \leftarrow isgrandparent(x,y), \\
& parent(f(x,y),y)) \leftarrow isgrandparent(x,y)\}
\end{aligned}$$

(ii) $\forall x (bird(x) \rightarrow flies(x))$

(iii) $\exists x \forall y \forall z (person(x) \wedge ((likes(x,y) \wedge y \neq z) \rightarrow \neg likes(x,z)))$

(iv) $\forall x ((\forall y\, IsDescendant(y,x)) \rightarrow Root(x))$

Exercise 2.3
Determine whether the following are valid inferences in first-order logic using resolution:

(i) $\forall x (P(x) \rightarrow Q(x)) \vdash \forall y (\neg Q(y) \rightarrow \neg P(y))$

(ii) $\forall x (P(x) \rightarrow Q(x)) \vdash \forall x (\neg Q(x) \rightarrow \neg P(x))$

(iii) $\forall x (P(x) \rightarrow Q(x)), P(a) \vdash Q(a)$

(iv) $\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$

(v) $\forall x (P(x) \rightarrow Q(x)), \forall x (Q(x) \rightarrow R(x)) \vdash \forall x (P(x) \rightarrow R(x))$

Solution of item (i):
We have two first-order formulae and we want to show, by resolution, that one is a derivation (or, conclusion) obtained from the other. In this case,

1. $\forall x (P(x) \rightarrow Q(x))$
2. (conclusion) $\forall y (\neg Q(y) \rightarrow \neg P(y))$

In order to use resolution, we need to transform a set of first-order formulae into a set of clauses. That is, transform each formula into Prenex Normal Form, Skolemize, and rewrite as clauses in conjunctive normal form. These transformation steps are found in page 28 of Logic and Resolution chapter. An important thing to remember is that, the formula that we want to prove is negated - and, as a result of the resolution process, we obtain the empty clause.

1. $\forall x (P(x) \rightarrow Q(x))$
   $\forall x (\neg P(x) \vee Q(x)$
   $\neg P(x) \vee Q(x)$

2. $\neg[\forall y(\neg Q(y) \rightarrow \neg P(y))]$ (negated conclusion)
$\neg[\forall y(\neg(\neg Q(y)) \vee \neg P(y))$
$\neg[\forall y(Q(y) \vee \neg P(y))$
$\exists y \neg(Q(y) \vee \neg P(y))$
$\exists y(\neg Q(y) \wedge P(y))$
$\neg Q(a) \wedge P(a)$

The set of clauses are then:

1. $\neg P(x) \vee Q(x)$
2. (a) $\neg Q(a)$
   (b) $P(a)$

We can now apply resolution in order to prove the conclusion suing the above clauses.

3. $Q(a)$ [1, 2(b)]
4. $\Box$ [3, 2(a)]

Note that we substitute the variable x by the constant term a in order to make possible the application of the resolution rule (eliminating literals $\neg P(x)$ and $P(a)$ in the first step).

Note: In each of the following 4 exercises, you are required to represent the sentences in predicate calculus, using only those predicates which are necessary. Negate the conclusion and convert to clause form, Skolemizing when necessary. Finally, prove the unsatisfiability of the resulting set of clauses using resolution.

Exercise 2.4
Consider the following set of clauses:

$$\begin{aligned} S = \{&P(x, g(x)), \\ &\neg P(a, g(a)) \vee Q(f(x, y), g(y)) \vee \neg R(x, y), \\ &R(c, a), \\ &\neg Q(f(c, x), g(a))\} \end{aligned}$$

Translate this set of clauses to the Horn clause notation. Use SLD-resolution to show that the resulting set of clauses is inconsistent.

Exercise 2.5 - TO BE SUBMITTED TO ONE OF THE LECTURERS/ASSISTANTS
Consider the following set of clauses S:

$$\begin{aligned} S = \{&\neg P(x, f(y)) \vee \neg Q(y, a) \vee R(x), \\ &Q(g(b), w) \vee S(d), \\ &P(c, u), \\ &\neg R(v) \vee S(d), \\ &\neg S(z)\} \end{aligned}$$

with x, y, z, u, v, w universally qualified variables and a, b, c, d constants. Prove by means of binary resolution that the set of clauses S is inconsistent.

Exercise 2.6
Consider the following axioms:

1. Every student who makes good grades is brilliant or studies.
2. Every student who is a CS major has some roommate.

3. Every student who has any roommate who likes to party goes to Sixth Street.

4. Anyone who goes to Sixth Street does not study.

5. (Conclusion) If every roommate of every CS major likes to party, then every student who is a CS major and makes good grades is brilliant.

Note: Make "roommate" a two-argument predicate, i.e., $roommate(x, y)$ means that x is a roommate of y.

Exercise 2.7
Consider the following axioms:

1. Anyone who is on Sixth Street and is not a police officer has some costume.

2. No CS student is a police officer.

3. Every costume that is good is a robot costume.

4. For anyone, if they are on Sixth Street and are happy, then every costume they have is good or they are drunk.

5. (Conclusion) If no CS student is drunk and every CS student on Sixth Street is happy, then every CS student on Sixth Street has some costume that is a robot costume.

Exercise 2.8
Consider the following axioms:

1. Anyone who buys carrots by the bushel owns either a rabbit or a grocery store.

2. Every dog chases some rabbit.

3. Mary buys carrots by the bushel.

4. Anyone who owns a rabbit hates anything that chases any rabbit.

5. John owns a dog.

6. Someone who hates something owned by another person will not date that person.

7. (Conclusion) If Mary does not own a grocery store, she will not date John.

Note: 'has a red nose' can be a single predicate.

Exercise 2.9
'Schubert's Steamroller' (C. Walther, AAAI-84, p. 330.)
"Wolves, foxes, birds, caterpillars, and snails are animals, and there are some of each of them. Also there are some grains, and grains are plants. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves. Wolves do not like to eat foxes or grains, while birds like to eat caterpillars but not snails. Caterpillars and snails like to eat some plants. Therefore there is an animal that likes to eat a grain-eating animal."