Radboud University Nijmegen

Faculty of Science

Playing Quantum Tic-Tac-Toe

On the (In)Effectiveness of Quantum Strategies

Thesis BSc Computing Science

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Abstract

In this thesis, we propose a quantum version of Tic-tac-toe, which we argue is more general and natural than existing quantizations. For our definition of quantum Tic-tac-toe, we show that a player with a strategy using quantum techniques (the *Quantum Player*) has no advantage over a player with a non-losing classical strategy (the Classical Player). However, when both the Quantum Player and the Classical Player compete against an *imperfect* classical opponent — that is, a player who occasionally makes mistakes — the Quantum Player can achieve a significantly higher probability of winning than the Classical Player.

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1 : Introduction

When quantum computers become large enough, they will outperform normal computers in many areas. Incredible future achievements are (speculatively) attributed to quantum computers, from breaking encryption to revolutionizing medicine [5, 2]. However, if it really is such an incredible computer would it then also be able to beat us in a simple game like Tic-tac-toe? If we were to pit a quantum computer against a classical computer for a game of Tic-tac-toe, who would win? And if the quantum computer would win from the classical computer, by how much? What would playing Tic-tac-toe on a quantum computer even look like? In this thesis, we try to answer these questions. We transform Tic-tac-toe to "Quantum Tic-tac-toe" and we determine if a quantum computer has an advantage against a classical computer.

Quantum Tic-tac-toe has already been investigated by several authors. Most use a quantization of Tic-tac-toe proposed by [3]. In this paper, Allan Goff proposes a version of Tic-tac-toe where the quantum player is quite limited; players may make equally distributed superpositions on two squares at once, or they may do a measurement of one of the squares. This removes most quantum strategies and is very restrictive; why not allow for three squares? In addition, generally it is customary to delay measurement to the end of some quantum program. The motivation behind this definition is as a teaching tool for acquainting students with quantum computing and its different aspects [13]. There are other authors who show some advantage for quantum strategies in some version of Quantum Tic-tac-toe, but these are often a derivation of Allan Goff's definition or alternative restrictive construction [8, 7]. There exist other quantizations [11, 10, 12, 6], but these are all not as general and natural as we strive to construct.

Not being satisfied with existing quantizations, we propose a new quantization of Tic-tac-toe (section 3.1), which is aimed at being as general as possible and allowing for as much quantumness as possible, while making sure classical strategies carry over to the quantum version. We motivate this construction and explore some alterations to our definition to ensure we are working with a solid framework of quantizing Tic-tac-toe (sections 3.3, 3.5). We then prove that quantum strategies provide no improvement over their classical counterparts (section 3.4). So, even though people speculate that quantum computers will revolutionize the world, they still cannot defeat us in Tic-tac-toe.

In chapter 3, we assumed the opponent is a perfect player, a computer who can make no mistake. We as humans, however, might make a mistake every so often. In chapter 4, we study quantum strategies against an *imperfect* opponent. Even though quantum computers have no advantage over perfect classical players, we show that if the opponent is imperfect a quantum computer can take considerably more advantage of their mistakes, than a classical computer could. We first define what it means for a player to be imperfect and what it means to make mistakes (section 4.1). We then propose two quantum strategies one of which shows a significant advantage over an optimal classical strategy (sections 4.3, 4.4). This is shown by simulating quantum Tic-tac-toe, the code can be found on GitHub.

In conclusion, when quantum computer become large enough, they will not be able to defeat a simple computer in Tic-tac-toe, but they will be able to defeat a simple human player.

2 : Preliminaries

In this chapter, we explain quantum computing to such a degree that the rest of this thesis should be understandable. If the reader would like to get a deeper understanding we refer to [9] as the generally accepted basic textbook on quantum computing. For computer science students in particular, who would like a lighter introduction, we recommend [15] or [14]. All theory explained in this section can be found in either one of these sources.

We assume basic linear algebra and computer science knowledge in this chapter.

2.1 Basics

In classical computing we express states as bits, either 1 or 0. Mechanically, this could mean something like flowing electricity or not. On quantum computers, we do not look at streams of electrons, but rather at a single quantum system, such as an electron or ion, which we call a “qubit”. There are quantum computers using other physical implementations, like trapped ions or superconducting circuits, but these details are out of scope here.

To read whether the qubit is 0 or 1, we perform a measurement. In many physical realizations, this corresponds to measuring the spin of an electron in a certain direction. While “spin” is not literally a rotation — it is a quantum property that can be measured along different axes — thinking of “spin” as actual rotation can be conceptually helpful. In quantum computing, we generally focus on one axis of this spin, and we measure this spin which yields either “up” or “down” spin (corresponding to 0 or 1, respectively).

A qubit can be described as a linear combination of these two basis states ($|0\rangle$ and $|1\rangle$), and its state can be visualized as a point on the surface of a three-dimensional unit sphere, called the *Bloch sphere*.

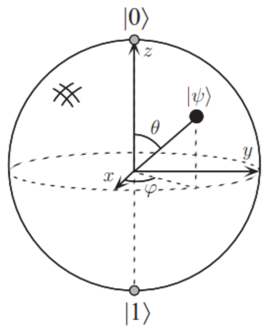


Figure 1: Bloch sphere representation of a qubit, [9]

We typically denote a qubit with a ψ or a ϕ, and we write as a vector in “bra-ket” notation: $\langle\psi|$ “bra”, $|\psi\rangle$ “ket”. We can write a qubit as a sum of basis element of $\mathbb{C}^2$, for which we choose the basis:

$$|0\rangle = (1, 0)^T$$

$$|1\rangle = (0, 1)^T$$

The concept of “up” and “down” is, of course, ambiguous, and it is dependent on the orientation of the measurement apparatus, mathematically we can see this as being dependent on the basis of the measurement. We can choose different bases for $\mathbb{C}^2$, as long as they are orthonormal. In this thesis, we only consider the standard basis. So, we write $|\psi\rangle$, some qubit, as a sum of basis elements:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}$$

Since we require $|\psi\rangle$ to be normalized, we get:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can also consider two qubit systems, which are unit-vectors in the space $\mathbb{C}^{2^2}$. The standard basis for this space is

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

We write $|00\rangle$ as shorthand for the tensor-product $|0\rangle \otimes |0\rangle = (1, 0, 0, 0)^T$. We can also look at systems with n qubits, which are unit-vectors in the space $\mathbb{C}^{2^2}$. Again the standard basis is similar:

$$\{|000\ldots0\rangle, |100\ldots0\rangle, |010\ldots0\rangle, \cdots, |111\ldots1\rangle\}$$

There are three main concepts in quantum computing we have to cover: measurement, superposition, and entanglement. We will cover each topic briefly.

- *Measurement.* When measuring some qubit, we are essentially measuring if the qubit spins up or down. In practice, this means that we pick some basis and measure the qubit for that basis. Quantum mechanics dictates that measurement makes the state collapse to a basis state, in this case, the qubit collapses to either one of the basis states. For this thesis, we consider the basis $\{|0\rangle, |1\rangle\}$, so measuring a qubit $|\psi\rangle$ makes it collapse to either $|0\rangle$ or $|1\rangle$. Important to understand is that when the spin is not completely "in-line" with a basis element, so not fully up or down, measurement is a probabilistic process. If we take some qubit:

  $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

  Then the probability of collapse to $|0\rangle$ is $|\alpha|^2$, and similarly it holds that collapse to $|1\rangle$ occurs with chance $|\beta|^2$. This also shows why it is important that qubits are normalized.

- *Superposition.* Mathematically, this is an easy concept. A state being in a superposition simply means it is a linear combination of $|0\rangle$ and $|1\rangle$. Notice that being in a superposition depends on the basis chosen. Conceptually, it is important to understand that if we have some

  $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

  with $\alpha, \beta \neq 0$, then this $|\psi\rangle$ is, in some sense, simultaneously $|0\rangle$ and $|1\rangle$. Now by performing operations on $|\psi\rangle$ we operate on both $|0\rangle$ and $|1\rangle$ simultaneously. We can also have superpositions among more qubits, for example a 2 qubit superposition could look like so:

  $$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \lambda|1\rangle) = \alpha\gamma|0\rangle|0\rangle + \alpha\lambda|0\rangle|1\rangle + \beta\gamma|1\rangle|0\rangle + \beta\lambda|1\rangle|1\rangle$$

  It must be remarked that this is not the most general way of writing a superposition between two qubits, it assumes the superposition can be rewritten as a product of two qubits, however in general this does not need to be the case. In cases where this is not possible we call it "entanglement".

- *Entanglement.* Lastly, we consider entanglement. Until now this was all able to be efficiently simulated by a classical computer [4]; but not any more. The previous example of a superposition of two states was able to be written as a product of two separate states:

$$\begin{aligned}|\psi\rangle &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \\ &= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) = |\psi_0\rangle \otimes |\psi_1\rangle\end{aligned}$$

  We call a state entangled if this separation is impossible. An example of such a state is:[1]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

  Conceptually, we can think of some magical if-condition binding these two qubits. The first qubits is $|0\rangle$ if and only if the second qubit is $|0\rangle$ and the same holds for $|1\rangle$.

2.2 Gates

Now that we have the fundamental concepts we move on to the workings of a quantum computer. Generally, in a quantum computer we work with more than one qubit of course. We manipulate these qubits with so-called quantum gates. The important part to remember about these gates is that they must be unitary, which implies they are norm preserving and invertible. A basic example of quantum gates for one qubit are the so-called "Pauli matrices":

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice that σ_x is essentially a NOT gate. We can also apply gates to multiple qubits, for example we could think of applying an AND-gate. The problem with a classical AND-gate is that it is not reversible and thus not being unitary. The solution is to make the AND-gate work on the two input qubits and have an extra output qubit.

Figure 2: Reversible AND gate

Taking $c = 0$ give a proper reversible AND gate.

In the context of quantum gates, there are a couple of things important to know for this thesis. First, we can make something called "controlled" gates. Since do not want to measure a qubit to see whether we want to apply some gate, since this would collapse the superposition, we use controlled gates. A controlled gate is defined on the basis states and only affects a qubit if the control qubit is in state $|1\rangle$. An example is the controlled-NOT gate, or CNOT:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

[1] This state is called the Bell state.

We can show its action by applying it to some basis states of $\mathbb{C}^{2^2}$:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}|00\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle$$

Nothing happens in this case, since the first qubit in $|00\rangle$, the control bit, is $|0\rangle$. If we apply the CNOT gate to $|10\rangle$, we get

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}|10\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |11\rangle$$

We can also represent gates in a circuit-like representation. The CNOT gate as a circuit is illustrated like so:

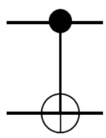


Figure 3: CNOT gate [9]

Here the bottom qubit is switched iff the top qubit is a $|1\rangle$.
In circuits, we also have a notation for measurement:

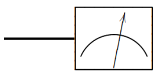


Figure 4: Measuring in circuits [9]

Note that measurement is not a quantum gate, since it is not reversible. Sometimes qubits get measured, but then it is indicated with a black dot at the end of the line, this basically means the qubit does get measured, but we do not care about the results.

Figure 5: Measuring disregarded result

The last gate we would like to consider is the "Toffoli gate", it is basically a controlled-CNOT gate or a CCNOT. Notice that the AND gate in fig. 2 is precisely a CCNOT gate. The Toffoli gate can also be used to implement a NAND gate:

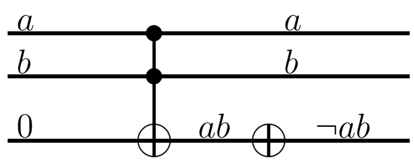


Figure 6: NAND gate using Toffoli gate

And also a FANOUT gate:

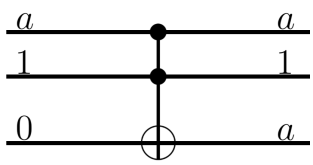


Figure 7: FANOUT gate

In this figure, the second qubit is referred to as an “auxiliary qubit”, these are qubits which are used kind of like scrap paper. Since all classical algorithms can be expressed using NAND and FANOUT gates we can express any classical algorithm using Toffoli and NOT gates, and enough auxiliary qubits ([9], section 1.4.1). This is an important result which we will use in many instances in this thesis.

3 : Tic-Tac-Toe

Tic-tac-toe is a simple two-player game for which both players have a non-losing strategy[2]. As a consequence, any game played in a classical setting — that is, on a classical computer or simply on a piece of paper — will always result in a draw (assuming both players play optimally).

In this chapter, we will define Quantum Tic-tac-toe (QT3) and show that even if we allow quantum techniques (e.g., superpositions) in a player's strategy that they cannot obtain an advantage over a player limited to classical strategies. That is, games between a quantum and a classical player still always end in a tie, assuming both play optimally.

We first focus on constructing both the classical and quantum formulations of Tic-tac-toe (in sections 3.1 and 3.2). In our quantization, we focus on defining Quantum Tic-tac-toe in a generalized and natural way — placing as few restrictions on moves and allowing for maximal "quantumness" — while ensuring that classical strategies remain valid. In particular, QT3 must be a generalization of classical Tic-tac-toe.

With this definition in place, we prove a quantum player cannot obtain advantage over a classical player (section 3.4). To further support our definition, we show that changing the time of measurement does not change the conclusion of section 3.4 (section 3.5).

3.1 Classical Tic-Tac-Toe

A Tic-tac-toe board is a 3-by-3 grid where each cell can be marked either X, O or be left unmarked. We number each cell on a Tic-tac-toe board as follows.

0	1	2
3	4	5
6	7	8

We now want to represent a specific board by storing the specific cells which are marked with either X or O. We represent one such marked field with e.g. $[0_X]$. Since there are 18 different cell-number–marking combinations, we can see a board as an element of the group $(\mathbb{Z}/2\mathbb{Z})^{18}$ with the standard addition modulo two and the following generating set:

$$
\begin{aligned}
(10000\ldots0) &= [0_O] \\
(11000\ldots0) &= [0_X] \\
(00100\ldots0) &= [1_O] \\
(00110\ldots0) &= [1_X] \\
&\vdots \\
(000\ldots0011) &= [8_X]
\end{aligned}
$$

[2]See e.g. Code, it identifies the empty board as a board where player X can force a tie and for each move X makes O can force a tie. Therefore, there exist non-losing strategies, see section 4.2 for more details.

Or in a more general form, we can write an element of the generating set as:

$$(\underbrace{0\ldots0}_{2n}\,1\,\chi_{\{X\}}(M)\,\underbrace{0\ldots0}_{16-2n-2}) = [n_M], \qquad \text{for } n = 0,\ldots,8,\ M \in \{X, O\}$$

where χ is the characteristic function.

We realize that this is a rather abstract and formal way of representing Tic-tac-toe boards, but a formal definition of Tic-tac-toe is necessary to be able to generalize the game to Quantum Tic-tac-toe.

Each element in this generating set represents a single field with a particular marking, which we will call a **field-marking**. We generally denote a marker, X or O, by M. We use $\overline{M}$ to represent the opponent's marker, so if $M = X$ then $\overline{M} = O$, and vice versa.

By defining field markings as elements in the group $(\mathbb{Z}/2\mathbb{Z})^{18}$. We can create a board by summing different elements of the generating set. For example, $\psi = [0_X] + [4_O]$, represents the board:

X		
	O	

Note that every valid board is in this group, but not every element in the group represents a valid board. For example, we can add elements of the generating set together to get a "nonsensical" boards, e.g. $[1_X] + [1_O]$. Hence, we will introduce the notion of *valid boards.*

Definition 3.1. Let ψ be an element in $(\mathbb{Z}/2\mathbb{Z})^{18}$. We call ψ a **valid board** if there exist distinct $n_0, \ldots, n_m \in \mathbb{N}_{\leqslant 8}$ and markers $M_0, \ldots, M_m \in \{X, O\}$ for some $0 \leqslant m < 9$, such that $\psi = \sum_{i=0}^{m} [(n_i)_{M_i}]$.

This definition says that a valid board can be written as a sum of field-markings — generating set elements — where each field in the grid is marked at most once. By enforcing that each field can only be marked at most once and restricting this sum to the elements in the generating set we see that we can only create boards which make "sense" — that is, they make sense to the extent that we can draw the board they represent. We remark here that not all valid boards can occur in natural gameplay. Consider this board:

$$\psi = [1_X] + [2_O] + [5_X] + [6_X] + [7_X]$$

By definition this is a valid board and as such we can draw it:

	X	O
		X
X	X	

The problem with this board is that it cannot occur in natural game play. The player playing as X has made too many moves here. In definition 3.5 we solve this problem by defining *valid reachable boards*. However, we first focus our attention on some other definitions.

Definition 3.2. Let $\psi \in (\mathbb{Z}/2\mathbb{Z})^{18}$ be a valid board. Let S be a set of field-markings. We call S the **construction of** ψ, if $\psi = \sum_{s \in S} s$.

Remark. S is unique. This follows from the set of field-markings being a minimal generating set of $(\mathbb{Z}/2\mathbb{Z})^{18}$ and S being a set of distinct field-markings.

Definition 3.3. Let $\psi \in (\mathbb{Z}/2\mathbb{Z})^{18}$ be a valid board. Let S be the construction of ψ. The board ψ is called **finished** if one of the following conditions holds:

0	1	2
3	4	5
6	7	8

1. There exists $[k_M], [l_M], [m_M] \in S$ such that $(k, l, m) \in$

   $\{(0, 1, 2), (3, 4, 5), (6, 7, 8), (0, 3, 6), (1, 4, 7), (2, 5, 8), (0, 4, 8), (2, 4, 6)\}$

   In this situation we write ψ is **finished with winner** M.

2. $|S| = 9$.

If condition 1 does not hold and condition 2 does we write ψ is **finished with a tie**.

Remark. Following the definition of a valid board, the definition above allows for both players to be denoted as winners. However, valid reachable boards, as will be defined later (def. 3.5), will only have at most one winner. Thus, we will keep talking about "the winner" of a board and not the "winners".

In the example below we illustrate three finished boards which adhere to conditions 1 and/or 2.

Condition 1 applies, winner: O

X	X	O
X	O	X
O		

Condition 1 and 2 apply, winner: X

X	O	X
X	X	O
O	O	X

Condition 2 applies, winner: Tie

X	X	O
O	O	X
X	O	X

We now define a *classical move for player M* as a mapping from boards $\psi \xrightarrow{\gamma} \phi$. Where a valid classical move is defined as:

Definition 3.4. Let $\psi,\ \phi\ \in (\mathbb{Z}/2\mathbb{Z})^{18}$ be valid boards. Let S, R be the constructions of ψ and ϕ, respectively. Let $\gamma \in \{[n_M] |\ n = 0 \dots 8,\ M = O, X\} \cup \{\bullet\}$. Then we write $\psi \xrightarrow{\gamma} \phi$, we say that M can move from ψ to ϕ, and we call γ a **valid classical move** for M from ψ to ϕ, if:

$$\begin{cases} S \subseteq R \text{ and } R \backslash S \equiv \{[n_M]\} & \text{if } \psi \text{ is not finished and } \gamma = [n_M] \\ \psi = \phi & \text{if } \psi \text{ is finished and } \gamma = \bullet \end{cases}$$

So, a move is valid when it either marks exactly one unmarked field with X or O, or if the board is in a *finished* state the move leaves the board unchanged. By defining valid moves in this way we ensure that there will always be only one winner on a specific board reached by valid moves.

We allow making moves, which do not change the game, when the board is finished. This means that games can continue indefinitely, even though the outcome of the game is already determined and unable to be changed. We choose to introduce this move to better facilitate quantization. We further motivate this decision in section 3.2.

We do not formally prove that this extension does not change gameplay, but the claim is rather intuitive. Consider an arbitrary game, once a win-state is determined — X, O or a tie — we allow for these "$\bullet$-moves". The moves do not change the outcome, and since they are only allowed *after* a winner is determined, they do not alter the outcome, nor the strategic decisions by either player.

As we discussed before, not all valid boards are boards which can occur in a game. Only valid boards, which can be reached by valid moves for alternating players X and O (with X the starting player), can occur in gameplay. We will now introduce $\mathcal{B} \subseteq (\mathbb{Z}/2\mathbb{Z})^{18}$ the set of all valid reachable boards.

Definition 3.5. Let $\psi \in (\mathbb{Z}/2\mathbb{Z})^{18}$ be valid board. Let $\psi_0 = 0 \in (\mathbb{Z}/2\mathbb{Z})^{18}$ be the empty board. Then if there exists a sequence $(\psi_1, \dots, \psi_n := \psi) \subset (\mathbb{Z}/2\mathbb{Z})^{18}$ of valid boards, for $n = |S|$ with S the construction of ψ, such that $\psi_i \xrightarrow{[n'_{M_i}]} \psi_{i+1}$ is a valid classical move for M_i, for $i = 0, \dots, n-1$, $n' \in \{0, \dots 8\}$, and:

$$M_i = \begin{cases} X, & \text{if } i \text{ even} \\ O, & \text{if } i \text{ odd} \end{cases}$$

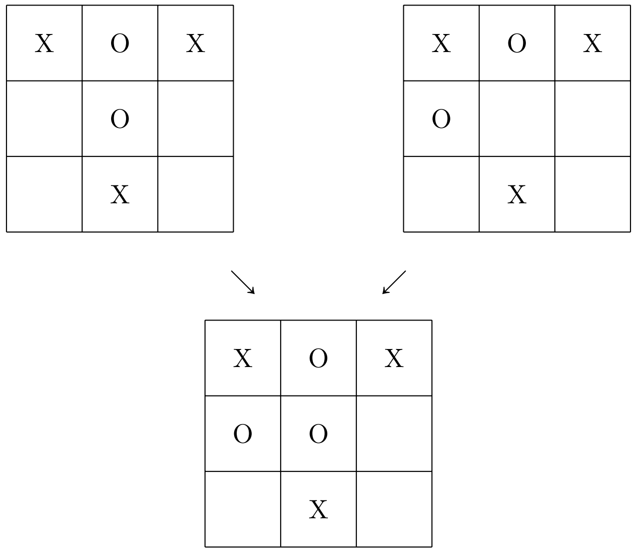
Then we call ψ a **valid reachable board**. We write $\mathcal{B} \subseteq (\mathbb{Z}/2\mathbb{Z})^{18}$ for the set of all valid reachable boards.

In other words, we simply ensure that for each valid reachable board ψ we have some "path" of valid moves by alternating players (starting at the empty board), which lead to ψ. This ensures ψ is reachable in ordinary gameplay.

3.2 Quantum Tic-Tac-Toe

Now that we have defined notation for classical Tic-tac-toe we can start making a quantum version of it. To do this we have to look at two things: quantum boards and quantum moves.

Before we define quantum boards, we first need to consider a problem with classical games. Since we will be playing Tic-tac-toe on a quantum computer we will need to translate classical moves to quantum moves, specifically, they will need to be represented by unitary matrices. This unitary requirement comes with a problem. Consider the following example:



Here we see that two *distinct* valid reachable boards can move to the *same* valid reachable board. Later on, we will define a quantum version of classical moves (def. 3.10). For now, we can see a "classical quantum move" as a "quantum version" of a map, which maps each valid reachable classical board to a next one — this mapping must adhere to classical valid moves. This map can effectively be seen as the "tactic" of a player (we define the notion of a *playable tactic* later in definition 3.7).

We need to consider a map with such an extensive domain — *all* valid reachable boards — since we will apply quantum moves to a superposition of boards. We can not know what boards are in this superposition, thus we must design some move which is valid for all boards. This will ensure that, later, quantum moves can adhere to the classical rules of Tic-tac-toe.

In the above scenario, we can see a tactic be applied to two distinct boards and both map to the same board. So a classical player playing according to this tactic would play from either one of the top boards to the bottom one. Since this tactic has the same output for different inputs we get that a tactic does not need to be *injective*[3].

To convert such a tactic to a "quantum move", we must be able to represent it with a unitary matrix. This means that it must be reversible and thus injective, which it does not have to be. For this reason, we need to change the way we represent boards in Quantum Tic-tac-toe to ensure that such a mapping can be reversible.

In this thesis, we propose adding *records* of moves to the boards. This record of moves is similar to move transcripts in chess, where each specific move can be encoded and recorded. With these transcripts we can see that a case similar to the example above cannot occur in chess games (paired with transcripts).

By adding records we now do not need the current board any more, since having all the moves made is enough information to construct the current board. However, we choose to keep the current board since this makes it notationally and conceptually easier[4]. In addition, only measuring the board — so not also records of moves — allows for more "quantumness". Furthermore, notice that in Tic-tac-toe having a record of moves would not provide an advantage nor would it change a player's gameplay, just

[3]We could alternatively define a "move one tactic" and a "move two tactic", etc. However, using one big tactic is easier to work with and is not less general.

[4]Remark that we are not concerned with space efficiency.

like in chess. We do not prove this here, but we can imagine that knowing the order of which prior moves were made in Tic-tac-toe has no effect on you next moves.

There are alternative solutions to this problem, e.g. recording past boards, but the proposed solution does not affect gameplay, allows for more "quantumness", and is notationally very easy. Since these are the main goals of our definition of QT3, we choose to use this solution.

Formally we define the new set of boards with records as follows.

Definition 3.6. Let $\mathcal{B}_{rec}$ be the set of **boards with records**.

$$\mathcal{B}_{rec} := \{(\psi, R_\psi) \in \mathcal{B} \times \bigcup_{n=0}^{10} (\{[n_M] | n = 0 \dots 8, M = O, X\} \cup \{\bullet\})^n | \text{ with } R_\psi \text{ a valid record } \}$$

A record R_ψ is considered valid if:

$$0 \overset{(R_\psi)_0}{\to} \psi_1 \overset{(R_\psi)_1}{\to} \dots \overset{(R_\psi)_{n-1}}{\to} \psi_n = \psi, \quad \wedge \quad |R_\psi| = n$$

are all valid moves and $\psi_i \overset{(R_\psi)_i}{\to} \psi_{i+1}$ is a valid classical move for player X if i is even, if i is odd it should be a valid classical move for player O.

Remark. Limiting these boards to a maximum of 10 total moves is sufficient for playing Tic-tac-toe. More moves can be played, but all of these will be "$\bullet$-moves".

A mapping from $\mathcal{B}_{rec}$ to $\mathcal{B}_{rec}$ according to classical moves, a tactic, can not, in general, be chosen to be injective. However, if we limit the total number of moves allowed to make to some N it is possible. We denote the sets of boards with at most N moves like so:

$$\mathcal{B}_{rec<N} := \{(\psi, R_\psi) \in \mathcal{B}_{rec} \mid |R_\psi| < N\}$$

Notice that $\mathcal{B}_{rec} = \mathcal{B}_{rec<11}$.

To turn a "tactic" into a quantum move we have can not consider any map from $\mathcal{B}_{rec}$ to $\mathcal{B}_{rec}$ we must have a map, which adheres to valid classical rules on $\mathcal{B}_{rec<10}$ and is a bijection, we call these *playable tactics*.

Definition 3.7. Let $\omega : \mathcal{B}_{rec} \to \mathcal{B}_{rec}$ be a map. Then we call ω a **playable tactic** if the following conditions hold:

1. ω is a bijection
2. $\forall (\psi, R_\psi) \in \mathcal{B}_{rec<10}$ with $(\phi, S_\phi) := \omega(\psi, R_\psi)$ we have:

$$\psi \overset{\gamma}{\to} \phi, \quad \text{and } R_\psi \| \gamma = S_\phi$$

   So, ω adheres to valid classical moves.

We denote the set of playable tactics by Ω.

We will now show that any "tactic" a player can think of can coincide with a playable tactic on the first ten moves. This immediately also gives that Ω is not empty.

Corollary 3.7.1. *For any mapping $f : \mathcal{B}_{rec} \to \mathcal{B}_{rec}$ (informally tactic) with $\forall (\psi, R_\psi) \in \mathcal{B}_{rec}$ with $(\phi, S_\phi) := \omega(\psi, R_\psi)$ we have:*

$$\psi \overset{\gamma}{\to} \phi, \quad \textit{and } R_\psi \| \gamma = S_\phi$$

There exists a playable tactic which coincides with f on $\mathcal{B}_{rec<10}$.

Proof. Take some such a map f as defined above. Notice that this f is well-defined since γ may be $\bullet$ and therefore f can be defined also on finished boards.

We will show that we can choose some other tactic which, is injective and coincides with f on $\mathcal{B}_{rec<10}$.

First, define the following map:

$$\omega : \mathcal{B}_{rec<10} \to f(\mathcal{B}_{rec<10}), \qquad \text{with } \omega \equiv f_{|\mathcal{B}_{rec<10}}$$

By definition we get that ω is surjective, and that ω coincide with f on $\mathcal{B}_{rec<10}$. We now show injectivity of ω. Take some $(\psi, R_\psi), (\phi, S_\phi) \in \mathcal{B}_{rec<10}$ with $\omega(\psi, R_\psi) = \omega(\phi, S_\phi) = (\kappa, Q_\kappa)$.

By definition of f we get:

$$R_\psi \| (Q_\kappa)_n = S_\phi \| (Q_\kappa)_n = Q_\kappa$$

With $n = |Q| - 1$. Which gives that $R_\psi = S_\phi$. Since records directly define the board in a board-record pair we also get that:

$$0 \overset{(R_\psi)_0}{\rightarrow} \psi_1 \overset{(R_\psi)_1}{\rightarrow} \psi_2 \overset{(R_\psi)_2}{\rightarrow} \cdots \overset{(R_\psi)_{n-1}}{\rightarrow} \psi = \phi \overset{(S_\phi)_{n-1}}{\leftarrow} \cdots \overset{(S_\phi)_2}{\leftarrow} \phi_2 \overset{(S_\phi)_1}{\leftarrow} \phi_1 \overset{(S_\phi)_0}{\leftarrow} 0$$

Which gives us that $(\psi, R_\psi) = (\phi, S_\phi)$ and thus ω is injective.

Now we want to extend ω to a domain of $\mathcal{B}_{rec}$. Since both $\mathcal{B}_{rec<10}$ and $\mathcal{B}_{rec}$ are finite and ω is a bijection we can extend ω to $\omega' : \mathcal{B}_{rec} \to \mathcal{B}_{rec}$ a bijection. Moreover, this ω' coincides with ω and therefore f on $\mathcal{B}_{rec<10}$. □

Strictly speaking, this is not yet sufficient for creating a unitary matrix representation of the playable tactic (see corollary 3.10.1). Before we can discuss this we first have to discuss quantum boards.

We denote the basis of the space of *quantum boards* by $|\mathcal{B}\rangle := \{|\psi\rangle \mid \psi \in \mathcal{B}_{rec}\} \subset \mathbb{C}^{2^{18\cdot 11}}$ notice that all quantum boards are valid and reachable (Also note that we can consider this as a subset of $\mathbb{C}^{2^{18\cdot 11}}$ since we can consider $\mathcal{B}_{rec}$ as a subset of $2^{18\cdot 11}$, 18 bits for the current board and 18 bits for each recorded/to be recorded move). We denote the space of all quantum boards by $\langle\mathcal{B}\rangle := \{\ |\psi\rangle \in \text{Span}_{\mathbb{C}}(|\mathcal{B}\rangle) \mid \|\,|\psi\rangle\,\| = 1\}$, all unit vectors in $\text{Span}_{\mathbb{C}}(|\mathcal{B}\rangle)$.

Furthermore, in this thesis, we use the following definition of the support of a vector:

$$Supp_B(v) = \{b_i \in B \mid c_i \neq 0\} \qquad \text{where } v = \sum_i c_i \cdot b_i$$

The set of basis elements with non-zero associated coefficients. We will omit the basis in notation when talking about the basis $|\mathcal{B}\rangle$, so $Supp \equiv Supp_{|\mathcal{B}\rangle}$.

For the remainder of this section we will write $|\psi R_\psi\rangle \in \langle\mathcal{B}\rangle$ as the superposition of boards and records which can be explicitly written like so:

$$|\psi R_\psi\rangle = \sum_{|\phi R_\phi\rangle \in Supp(|\psi R_\psi\rangle)} \alpha_\phi |\phi R_\phi\rangle, \quad \alpha_\phi \in \mathbb{C}$$

We now provide the definition of *quantum moves*, which are applied on a superposition of quantum boards, their records and some auxiliary qubits.

Definition 3.8. Let $U \in M_{(18\cdot 11+n)\times(18\cdot 11+n)}(\mathbb{C})$ be a unitary matrix. We call U a **valid quantum move** if for each $|\psi R_\psi\rangle \in |\mathcal{B}\rangle$ we have $U|\psi R_\psi 0^n\rangle = |\psi' R'_{\psi'} 0^n\rangle$ for some $|\psi' R'_{\psi'}\rangle \in \langle\mathcal{B}\rangle$, and, moreover, for each $|\phi R_\phi 0^n\rangle \in Supp(U|\psi R_\psi 0^n\rangle)$ we have $\psi \overset{\gamma}{\rightarrow} \phi$ is a valid classical move for some player $M \in \{X, O\}$.

The set of all valid quantum moves is denoted by $\mathbb{U}$.

Remark. As is customary, we generally omit the auxiliary qubits from notation. In addition, we might sometimes omit the record in notation.

Remark. We do not require that the quantum move is valid for some player, rather we assume a quantum move is valid for *each* quantum board. This means it is not necessary to specify validity for some player since the validity is defined in terms of classical moves, which act on classical Tic-tac-toe boards, for which the player-to-move can be determined. This means that by the above definition a valid quantum move can be applied to any quantum board and performs moves specified according to the player-to-move on a board *not* the player applying the move.

So if we apply a valid quantum move to a superposition of boards we require that applying the move to each board separately provides a number of valid classical moves.

Our definition of quantum moves is the main reason why our definition of QT3 is more general and natural; we do not put any arbitrary restrictions on the design of a quantum move. As remarked above, it imposes no arbitrary restrictions on quantum moves. The only constraint is that a quantum move must respect the classical structure of the game, ensuring that the classical game is naturally embedded in the quantum version. This makes our definition more general and conceptually natural than those proposed in earlier works [3, 8, 11, 6]. These authors, generally, only permit making only equal superpositions over two cells [3, 11] or over n cells [6]. Our approach avoids such artificial constraints, leading to a more general and flexible quantization.

3.2.1 Playing the Quantum Game

To play QT3, both players get assigned either the marker X or O. Each player will be playing several moves in this game. For each move, the players create a separate unitary which needs to be a valid quantum move for their marker, as described in definition 3.8. The set of these chosen unitaries will be called a *strategy*. All the unitaries/moves will be interlaced, one after the other, and implemented as a quantum circuit. The board will be initialized to the starting state of only zeros. All unitaries/moves will be applied, after which the first 18 qubits will be measured. Important to emphasize here is that we only measure the *final* board and not the entire the record. The board upon which the final superposition collapses will be the resulting board. From that board, the winner will be determined like in classical Tic-Tac-Toe (definition 3.3)[5].

Our choice to defer all measurement until the *end* of the game, we believe, is a more natural way of quantizing Tic-tac-toe. Most of the earlier works also permit intermediate measurement of specific cells [3, 8, 11, 6]. Deciding the winner by collapsing the final superposition is also more general than the method proposed by [7], where the winner is decided to be the first player with a straight line of thee cells with a total squared absolute amplitude of at least 3. We believe measuring to get a final board is more in-line with quantum computing conventions.

Traditionally, the player with marker X, from now on called *Player X*, plays 5 moves and *Player O* plays 4 moves. Since by our definition, a move on a finished board has no effect (classically) we will allow both players to make 5 moves, for ease of notation. This is not the only reason behind allowing moves on finished boards. We also allow marking moves on finished boards, because, when playing Quantum Tic-tac-toe, it is possible that not all boards will be finished at the same time. Consider the example game below (figure 8). In this example game, we see that some boards are finished after only eight moves while others after nine moves. If not all boards in the superposition are finished we will still need to be able to apply moves on the entire superposition to properly finish

[5] The impact of measuring and *then* determining the winner will be discussed later (see section 3.5).

the quantum game. Since, applying a move on a superposition with finished board also means that we need to be able to apply some classical move on the finished boards, we allow players to play a move not affecting the game.

We now discuss some notation and naming conventions used in this thesis.

Let us denote the valid quantum moves prepared by player X as $X_1, \ldots, X_5$ and the moves prepared by player O as $O_1, \ldots, O_5$. The starting state of the game is always $|0^{11 \cdot 18}\rangle$, from now on denoted by $|0\rangle$. Let $|\psi\rangle$ be a board occurring in the game played with these moves. Suppose $|\psi\rangle$ is of the form:

$$|\psi\rangle = O_n X_n O_{n-1} X_{n-1} \ldots O_1 X_1 |0\rangle, \quad 0 \leqslant n \leqslant 5$$

We now say that board $|\psi\rangle$ "has been made with $2n$ moves", "has $10 - 2n$ moves left to be played", and "it is player X's turn", or "player X is to move". Similarly, you can get:

$$|\psi\rangle = X_n O_{n-1} X_{n-1} \ldots O_1 X_1 |0\rangle, \quad 0 \leqslant n \leqslant 5$$

Here we say that board $|\psi\rangle$ "has been made with $2n - 1$ moves", "has $10 - (2n - 1)$ moves left to be played", and "it is player O's turn", or "player O is to move".

We now introduce a function, which, given a superposition of finished boards, gives a vector in the space spanned by $\{X, O, T\}$. This vector represents the probabilities of having either X, O, or T as the outcome of the game upon measurement.

Definition 3.9. We define $\mathcal{O} : \mathcal{F} \to \text{Span}_{\mathbb{R}}(\{X, O, T\})$ called the **outcome function**. Here $\mathcal{F} := \{|\psi\rangle \in \langle\mathcal{B}\rangle \mid \phi \text{ a finished board }, \forall |\phi R_\phi\rangle \in Supp(|\psi\rangle)\}$ is the set of superpositions of finished boards. Let $|\psi R_\psi\rangle \in \mathcal{F}$. Then:

$$\mathcal{O}(|\psi R_\psi\rangle) := \begin{cases} X, & \text{if } |\psi R_\psi\rangle \in |\mathcal{B}\rangle \text{ and the winner of } \psi \text{ is denoted by X} \\ O, & \text{if } |\psi R_\psi\rangle \in |\mathcal{B}\rangle \text{ and the winner of } \psi \text{ is denoted by O} \\ T, & \text{if } |\psi R_\psi\rangle \in |\mathcal{B}\rangle \text{ and the winner of } \psi \text{ is denoted by T} \\ \sum\limits_{|\phi\rangle \in Supp(|\psi R_\psi\rangle)} |\alpha_\phi|^2 \mathcal{O}(|\phi\rangle), & \text{if } \sum\limits_{|\phi\rangle \in Supp(|\psi R_\psi\rangle)} \alpha_\phi |\phi\rangle \end{cases}$$

We can now define "classical non-losing strategy". First, we introduce the following orderings on $\{X, O, T\}$: $\geqslant_X$: $X \geqslant_X T \geqslant_X O$ and conversely: $\geqslant_O$: $O \geqslant_O T \geqslant_O X$. In addition, we define a valid *classical quantum move.*

Definition 3.10. Let U be some valid quantum move. Then U is called a **classical quantum move** if

$$\forall |\psi R_\psi\rangle \in |\mathcal{B}\rangle : U|\psi R_\psi\rangle = |\phi R_\phi\rangle \in |\mathcal{B}\rangle$$

The set of all classical quantum moves is denoted by $\mathbb{U}_C$.

So, a valid classical quantum move does not change amplitude nor create a new superposition. It acts like a normal classical move would.

Corollary 3.10.1. *Any playable tactic $\omega \in \Omega$ can be expressed as a classical quantum move.*

Proof. First, we show that any playable tactic can be expressed as a unitary matrix. Take some $\omega \in \Omega$ a playable tactic.

By definition of playable tactic (def. 3.7) we get that ω is a bijection. Notice that $\mathcal{B}_{rec}$ is not orthonormal, but $|\mathcal{B}\rangle$ — its quantum representation — is. Let $f : \mathcal{B}_{rec} \to |\mathcal{B}\rangle$ be the canonical mapping from any board to its quantum representation. We now get that $f^{-1} \circ \omega \circ f$ is a bijection on $|\mathcal{B}\rangle$.

We can expand this mapping to a linear operator F:

$$F(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha f^{-1} \circ \omega \circ f(\psi) + \beta f^{-1} \circ \omega \circ f(\phi)$$

By definition, F coincides with $f^{-1} \circ \omega \circ f$ on the basis $|\mathcal{B}\rangle$. Since this basis is orthonormal we also get that F preserves the inner product and F is still a bijection, which gives that F is a *unitary* operator.

Let U be the unitary matrix representation of F. We now prove that U is a valid quantum move, and it is a classical quantum move, which coincides with ω.

Take some $|\psi R_\psi\rangle \in |\mathcal{B}\rangle$. Then:

$$\begin{aligned} U|\psi R_\psi\rangle &= F(|\psi R_\psi\rangle) \\ &= f^{-1} \circ \omega \circ f(|\psi R_\psi\rangle) \\ &= f^{-1} \circ \omega(\psi R_\psi) \\ &= |\phi R_\psi \| \gamma\rangle \end{aligned}$$

With $\psi \xrightarrow{\gamma} \phi$, which gives $|\phi R_\psi \| \gamma\rangle \in |\mathcal{B}\rangle$. This means that U is a valid quantum move. Since the resulting position is not a superposition we can also directly conclude that U is a classical quantum move. Finally, by construction of U, we get that U coincides with ω. $\square$

So, all playable tactics for the classical game can be translated to the quantum game. This means that we can have a classical players play Quantum Tic-tac-toe. Moreover, this also shows that there exist quantum moves; $\varnothing \neq \mathbb{U}_c \subseteq \mathbb{U}$.

In this section, we want to classify boards based on their eventual outcome. We will now define sets of boards for which, given a classical strategy, we can be certain the result of the game will be in some $\mathcal{M} \subseteq \{X, O, T\}$.

Definition 3.11. Let $\vec{U} = (U_1, U_2, \ldots, U_n) \in \mathbb{U}_C^n$, $1 \leqslant n \leqslant 5$ a sequence of valid classical quantum moves with player $M \in \{X, O\}$ playing the strategy $\vec{U}$. Let $\mathcal{M} \subseteq \{X, O, T\}$ a subset of markings and let $|\psi\rangle \in |\mathcal{B}\rangle$ with player $M' \in \{X, O\}$ the payer-to-move. Also assume that $|\psi\rangle$ has the appropriate number of moves left to be played such that precisely all moves in $\vec{U}$ can be played. Then we say $\vec{U}$ guarantees $\mathcal{M}$ for M when applied to $|\psi\rangle$ with M' to play, when:

$$|\psi\rangle \in \mathcal{B}^{M,\vec{U}}_{M',\mathcal{M}} \text{ if } \begin{cases} \mathcal{O}(|\psi\rangle) \in \mathcal{M} & \text{if } |\psi\rangle \in \mathcal{F} \\ U^C|\psi\rangle \in \mathcal{B}^{M,\vec{U}'}_{\overline{M'},\mathcal{M}} & \text{if } M = M', U^C\vec{U}' = \vec{U} \\ \forall V^C \in \mathbb{U}_C : V^C|\psi\rangle \in \mathcal{B}^{M,\vec{U}}_{\overline{M'},\mathcal{M}} & \text{if } M \neq M' \end{cases}$$

So if $|\psi\rangle \in \mathcal{B}^{M,\vec{U}}_{M',\mathcal{M}}$ then if player M plays the strategy specified by $\vec{U}$ then the outcome of the final board of the game will always be in the set $\mathcal{M}$.

We can now define "classical non-losing strategy".

Definition 3.12. Let P be a valid classical quantum move. Let $\vec{P} = (P, P, P, P, P)$. Then P is called a **classical non-losing strategy** if:

$$|0\rangle \in \mathcal{B}^{M,\vec{P}}_{X,\{M,T\}}, \ \forall M \in \{X, O\}$$

Remark. Such a strategy exists, since the classical algorithm for a non-losing Tic-tac-toe strategy exists. This strategy can be made into a quantum algorithm using Tofoli-gates (as described in [9] section 1.4.1).

So for any opposing strategy if player M plays a classical non-losing strategy then if the game starts in board $|0\rangle$ the eventual outcome of the game is either winning for player M or a tie.

In the rest of this section, we consider a game between a classical non-losing strategy (P, P, P, P, P) the *classical strategy*, and $M_1^Q, \ldots, M_5^Q$ the *quantum strategy*.

3.2.2 Game Illustration

Player O is the classical player, this player plays a classical strategy, which coincides with a classical non-losing strategy. Player X, in turn, is the quantum player who plays a quantum strategy. Both players have prepared five unitaries (valid quantum moves) which are used to play the game. Player X adheres to the following strategy, described in pseudo-code below.

```
if (Board.emptyFields().length() >= 2 and !Board.isFinished())
    f1 = Board.emptyFields().pop()      // sorted from upper-left to bottom-right
    f2 = Board.emptyFields().pop()
    Board.mark(fields=[f1, f2], mark=X) // create equally distributed superposition
else if (Board.emptyFields().length() == 1 and !Board.isFinished())
    f1 = Board.emptyFields().pop()
    Board.mark(fields=[f1], mark=X)
else
    pass                                // do nothing, game is finished
```

Remark. We can implement this as a quantum circuit since we can encode any classical circuit as a quantum circuit.

Since starting from the empty board would provide a rather extensive illustration we offer a segment of a game where the first four moves led to $|\psi_1\rangle$ as in the figure below. This $|\psi_1\rangle$ does not follow from the strategies given above, but it makes a more interesting starting point for this example.

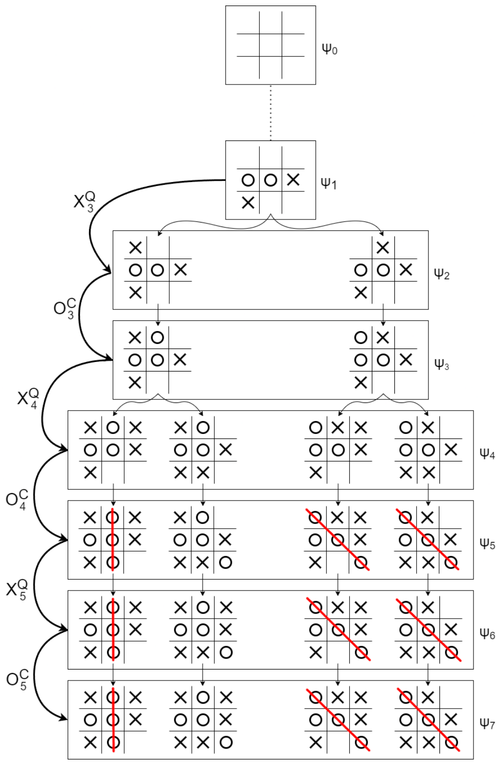


Figure 8: Illustration of an example quantum game

Now the game has been "played" with a resulting quantum position $|\psi_7\rangle$ which is a superposition of the four boards all with amplitude $\frac{1}{\sqrt{4}}$. Measurement of $|\psi_7\rangle$ collapses the superposition and gives one board as a result. The winner of this game is then the winner of this collapsed board, determined classically.

Let $|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ be the boards in the superposition $|\psi_7\rangle = \frac{1}{\sqrt{4}}(|\phi_0\rangle + |\phi_1\rangle + |\phi_2\rangle + |\phi_3\rangle)$, numbered left to right as in figure 8. Note that classically we can determine that board ϕ_2 is a tie (T) and ϕ_0, ϕ_2 and ϕ_3 are wins for player O.

$$
\begin{aligned}
\mathbb{P}(\text{Game is a tie}) &= \mathbb{P}(\text{Measuring board } \phi_1) = |\langle\psi_7|\phi_1\rangle|^2 = \frac{1}{4} \\
\mathbb{P}(\text{Player O wins}) &= \mathbb{P}(\text{Measuring either one of the boards } \phi_0, \phi_2, \phi_3) \\
&= \sum_{i=0,2,3} |\langle\psi_7|\phi_i\rangle|^2 = \sum_{i=0,2,3} |\frac{1}{\sqrt{4}}|^2 = \frac{3}{4} \\
\mathbb{P}(\text{Player X wins}) &= 0
\end{aligned}
$$

We can implement the game played as the following quantum circuit:

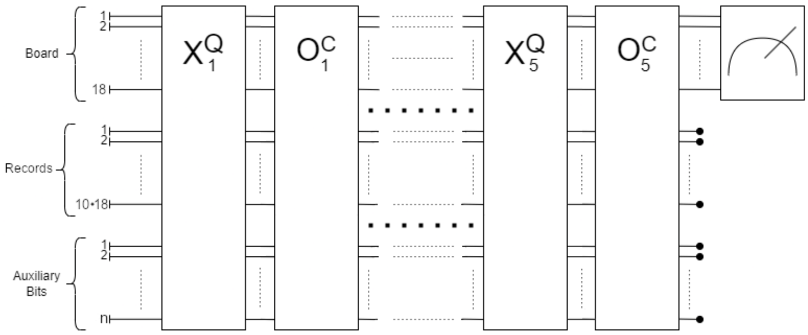


Figure 9: Quantum Tic-tac-toe circuit

3.3 Generalization of Classical Tic-Tac-Toe

To motivate why this construction of QT3 generalizes the classical game, we consider a scenario where both players are limited to making valid classical quantum moves. In this case, no superposition occurs — the game remains within the basis of quantum board space, $|\mathcal{B}\rangle$. Notice that each board in the basis has a one-to-one correspondence to a classical board in $\mathcal{B}_{rec}$. In addition, valid classical quantum moves are also analogous to their classical counterparts — meaning, for each classical quantum move $U|\phi\rangle = |\psi\rangle$, $U \in \mathbb{U}_C$ there is a corresponding classical move $\phi \xrightarrow{\gamma} \psi$. Finally, the validity of quantum moves is defined terms of the validity of classical moves, and the win condition on a quantum board is inherited from classical version. (Although not a proof, we hope this to be a convincing argument as to why the construction above should be considered a valid generalization of the classical game to quantum.)

We did introduce two changes to the classical Tic-tac-toe game to facilitate proper quantization. We introduced records to boards, and we allowed for moves to be made which do not change the board, iff the board is finished. As we have motivated at their definitions, these extensions of Tic-tac-toe do not change gameplay.

In addition to these two alterations we also decided to determine the win-state (the winner of a board or that the board is tied) *after* measurement. In section 3.5, we show that determining the win-state of board *before* measurement yields the exact same results as we will show in the coming section (3.4).

3.4 No Possible Improvement

It turns out that quantizing Tic-tac-toe does not change the gameplay for a classical player. Both the non-losing strategies and the outcomes of these strategies remain the same as in classical Tic-tac-toe. So a classical player has no reason to change the way they play Tic-tac-toe, even if the opponent is a quantum player.

We consider both assignments of markers to quantum and a classical player; the quantum player as X and the classical player as O, and vice versa.

However, first we prove the following lemma.

Lemma 3.13. *Let $|\psi\rangle \in \langle\mathcal{B}\rangle$ and let $U \in \mathbb{U}$ be an arbitrary quantum move, then:*

$$Supp(U|\psi\rangle) \subseteq \bigcup_{|\phi\rangle \in Supp(|\psi\rangle)} Supp(U|\phi\rangle)$$

Proof. Take some $|\phi_1\rangle, |\phi_2\rangle \in \langle\mathcal{B}\rangle$. We write:

$$|\phi_1\rangle = \sum_{|\kappa\rangle \in |\mathcal{B}\rangle} \alpha_\kappa |\kappa\rangle$$

$$|\phi_2\rangle = \sum_{|\kappa\rangle \in |\mathcal{B}\rangle} \beta_\kappa |\kappa\rangle$$

By construction of the support, we get that:

$$Supp(\gamma_1|\phi_1\rangle + \gamma_2|\phi_2\rangle) = \{|\kappa\rangle \in |\mathcal{B}\rangle \mid \gamma_1\alpha_\kappa + \gamma_2\beta_\kappa \neq 0\}$$

and

$$Supp(\gamma_1|\phi_1\rangle) \cup Supp(\gamma_2|\phi_2\rangle) = \{|\kappa\rangle \in |\mathcal{B}\rangle \mid \gamma_1\alpha_\kappa \neq 0 \vee \gamma_2\beta_\kappa \neq 0\}$$

From this it is clear that: $Supp(|\phi_1\rangle + |\phi_2\rangle) \subseteq Supp(|\phi_1\rangle) \cup Supp(|\phi_2\rangle)$. Since the basis and therefore the support of $|\psi\rangle$ is finite — with a finite number of applications of the above derivation — we get:

$$\begin{aligned} Supp(U|\psi\rangle) &= Supp(U \cdot \sum_{|\kappa\rangle \in Supp(|\psi\rangle)} \gamma_\kappa |\kappa\rangle) \\ &= Supp(\sum_{|\kappa\rangle \in Supp(|\psi\rangle)} \gamma_\kappa U|\kappa\rangle) \\ &\subseteq \bigcup_{|\kappa\rangle \in Supp(|\psi\rangle)} Supp(\gamma_\kappa U|\kappa\rangle) \end{aligned}$$

This concludes the proof. □

With this lemma, we can prove the following theorem. We use the following notation for indicating subsequences $(n_i)_j^\ell$, which is the subsequence $(n_j, n_{j+1}, \ldots, n_{\ell-1}, n_\ell)$, a subsequence of $(n_0, n_1, n_2, \ldots)$.

Theorem 3.14. *Let $|\psi\rangle \in \langle\mathcal{B}\rangle$ be a quantum board with player M to move. Let $\vec{U} = (U_1^C, \ldots, U_n^C), 1 \leqslant n \leqslant 5$, be a sequence of valid classical quantum moves. Then:*

$$Supp(|\psi\rangle) \subseteq \mathcal{B}_{M,\{M,T\}}^{M,\vec{U}} \Rightarrow Supp(U_1^C|\psi\rangle) \subseteq \mathcal{B}_{\overline{M},\{M,T\}}^{M,(U_i^C)_2^n}$$

and

$$Supp(|\psi\rangle) \subseteq \mathcal{B}_{\overline{M},\{M,T\}}^{M,\vec{U}} \Rightarrow \forall V^Q : Supp(V^Q|\psi\rangle) \subseteq \mathcal{B}_{M,\{M,T\}}^{M,\vec{U}}$$

Proof.

- *Statement 1:* Assume $Supp(|\psi\rangle) \subseteq \mathcal{B}_{M,\{M,T\}}^{M,\vec{U}}$. Take some $|\phi R\rangle \in Supp(U_1^C|\psi\rangle)$. We must show that $|\phi R\rangle \in \mathcal{B}_{\overline{M},\{M,T\}}^{M,(U_i^C)_2^n}$.

  Take some $|\kappa S\rangle \in Supp(|\psi\rangle)$ such that there exists some classical move, γ:

  $$\kappa \xrightarrow{\gamma} \phi$$

  This move exists by definition of valid (classical) quantum moves (def. 3.10). Moreover, this move coincides with the classical quantum move U_1^C, that is, $U_1^C|\kappa S\rangle = |\phi R\rangle$.

  By assumption $|\kappa S\rangle \in \mathcal{B}_{M,\{M,T\}}^{M,\vec{U}}$. By definition 3.11

  $$U_1^C|\kappa S\rangle = |\phi R\rangle \in \mathcal{B}_{\overline{M},\{M,T\}}^{M,(U_i^C)_2^n}$$

  Thus:

  $$Supp(U^C|\psi\rangle) \subseteq \mathcal{B}_{\overline{M},\{M,T\}}^{M,(U_i^C)_2^n}$$

- *Statement 2:* Assume $Supp(|\psi\rangle) \subseteq \mathcal{B}^{M,\vec{U}}_{\overline{M},\{M,T\}}$. Take some valid quantum move $V^Q \in \mathbb{U}$ arbitrarily. We must show that $Supp(V^Q|\psi\rangle) \subseteq \mathcal{B}^{M,\vec{U}}_{M,\{M,T\}}$. By lemma 3.13, it suffices to show that:

$$Supp(V^Q|\phi R\rangle) \subseteq \mathcal{B}^{M,\vec{U}}_{M,\{M,T\}}, \quad \forall |\phi R\rangle \in Supp(|\psi\rangle)$$

Take some $|\phi R\rangle \in Supp(|\psi\rangle)$. Take $|\kappa S\rangle \in Supp(V^Q|\phi\rangle)$, we must show that $|\kappa S\rangle \in \mathcal{B}^{M,\vec{U}}_{M,\{M,T\}}$. There exists some classical move, γ:

$$\phi \xrightarrow{\gamma} \kappa$$

This move exists by definition of valid quantum moves (def. 3.8).

By the assumption we get that $|\phi R\rangle \in \mathcal{B}^{M,\vec{U}}_{\overline{M},\{M,T\}}$ which implies that $|\kappa S\rangle \in \mathcal{B}^{M,\vec{U}}_{M,\{M,T\}}$. So we are done.

□

This theorem allows us to show that the outcome of any game is either a tie or a win for the classical player playing the classical non-losing strategy.

Corollary 3.14.1. *Let $\vec{M} = (P, P, P, P, P)$ be some classical non-losing strategy and let $\overline{M}^Q_1, \ldots, \overline{M}^Q_5$ be some quantum strategy, then:*

$$\begin{cases} \mathcal{O}(Supp(\overline{M}^Q_5 P \ldots \overline{M}^Q_1 P|0\rangle)) \subseteq \{M, T\} & \text{if } M = X \\ \mathcal{O}(Supp(P\overline{M}^Q_5 \ldots P\overline{M}^Q_1 |0\rangle)) \subseteq \{M, T\} & \text{if } M = O \end{cases}$$

Proof. By definition of P (3.12) we get:

$$|0\rangle \in \mathcal{B}^{M,\vec{M}}_{X,\{M,T\}}$$

By applying of the above theorem (3.14) 10 times we get:

$$\begin{cases} Supp(\overline{M}^Q_5 P \ldots \overline{M}^Q_1 P|0\rangle) \subseteq \mathcal{B}^{M,\vec{M}}_{X,\{M,T\}} & \text{if } M = X \\ Supp(P\overline{M}^Q_5 \ldots P\overline{M}^Q_1 |0\rangle) \subseteq \mathcal{B}^{M,\vec{M}}_{X,\{M,T\}} & \text{if } M = O \end{cases}$$

The desired conclusion follows. □

The following lemma states that for a player to have a chance of winning, there needs to be a board in the final superposition whose outcome (i.e. evaluated by the outcome function) is in favour of that player. If there is no such board, then the final superposition can never collapse to a board upon which the player is winning.

Lemma 3.15. *Let $|\psi\rangle \in \langle\mathcal{B}\rangle$ be a quantum board with 0 moves left to be made. Then if $Pr_M(\mathcal{O}(|\psi\rangle)) = 0$, where Pr_M is the standard projection function onto $M \in \{X, O\}$. Then $\mathbb{P}$(player M wins) $= 0$.*

Proof. First, we note that the probability of $|\psi\rangle$ collapsing to a board $|\phi\rangle \in |\mathcal{B}\rangle$ is $|\langle\psi|\phi\rangle|^2$.
So we get the following probability:

$$\mathbb{P}(\text{player M wins}) = \sum_{|\phi\rangle \in W_M} |\langle\psi|\phi\rangle|^2$$

where $W_M := \{|\psi\rangle \in |\mathcal{B}\rangle \mid \mathcal{O}(|\psi\rangle) = M\}$.

Let $|\psi'\rangle$ be an arbitrary board in W_M and assume $|\langle\psi|\psi'\rangle| = |\alpha| > 0$. Then we get:

$$Pr_M(\mathcal{O}(|\psi\rangle)) = \sum_{|\phi\rangle \in W_M} |\alpha_\phi|^2 \geqslant |\alpha|^2 > 0$$

which contradicts the assumption of $Pr_M(\mathcal{O}(|\psi\rangle)) = 0$. Thus, by contradiction, we conclude that for all $|\phi\rangle \in W_M$:

$$|\langle\psi|\phi\rangle|^2 = 0 \implies \mathbb{P}(\text{player M wins}) = \sum_{|\phi\rangle \in W_M} |\langle\psi|\phi\rangle|^2 = 0$$

which concludes the proof. □

So, to win as a quantum player, we need to find a strategy that gets us a favourable board in the final to-be-measured superposition.

With this final lemma and the previously stated corollary (3.14.1), we can finally state and prove the final theorem, which concludes that using a quantum strategy does not provide the player with additional benefits over a player who plays a classical non-losing strategy.

Theorem 3.16. *Let $M \in \{X, O\}$ be the quantum player and $\overline{M}$ the classical player. Then assuming $\overline{M}$ plays some classical quantum strategy coinciding with a non-losing strategy, then $\mathbb{P}(M \text{ wins}) = 0$.*

Proof. Let $|\psi\rangle$ be the quantum board after 10 moves, derived from the strategies chosen by M and $\overline{M}$. Corollary 3.14.1 gives:

$$\begin{aligned} &Supp(\mathcal{O}(|\psi\rangle)) \subseteq \{\overline{M}, T\} \\ \implies\ &Pr_M(\mathcal{O}(|\psi\rangle)) = 0 \\ \underset{\text{lemma 3.15}}{\implies}\ &\mathbb{P}(\text{player M wins}) = 0 \end{aligned}$$

This concludes the proof. □

So applying quantum techniques in some strategy does not give an advantage over a player who is limited to classical techniques. In the next section, we will discuss and show that changing the moment of measurement to *after* the win-state of each board in the superposition has been determined does not affect the results obtain in this section.

3.5 Effects of Postponing Measurement

A major decision made in the definition of quantum Tic-tac-toe was measuring the superposition of boards and *then* determine the winner. When we defined the "outcome function" 3.9 the following property was given:

$$\mathcal{O}(|\psi\rangle) = \mathcal{O}(\sum_{|\phi\rangle \in Supp(|\psi\rangle)} \alpha_\phi |\phi\rangle) = \sum_{|\phi\rangle \in Supp(|\psi\rangle)} |\alpha_\phi|^2 \mathcal{O}(|\phi\rangle)$$

We chose this specific definition of the outcome-function to coincide with probability of the superposition collapsing to a board in a certain outcome. In this section, we assume X, O and T are encoded with three (qu)bits namely:

$$\begin{aligned} X &= 001 \\ O &= 010 \\ T &= 100 \end{aligned}$$

We introduce a new unitary $\hat{\mathcal{O}}$. $\hat{\mathcal{O}}$ is the unitary implementation of the following function, which can be done with Toffoli gates:

$$f(|\psi\rangle|0^3\rangle) = f\left(\sum_{|\phi\rangle\in Supp(|\psi\rangle)} \alpha_\phi|\phi\rangle\right) = |\psi\rangle \sum_{|\phi\rangle\in Supp(|\psi\rangle)} \alpha_\phi \mathcal{O}(|\phi\rangle)$$

Note that f is injective. For this section we measure the 3 qubits after the board state, so qubits $18\cdot 11+1,\ 18\cdot 11+2,\ 18\cdot 11+3$. This means measurement after applying $\hat{\mathcal{O}}$ has the following property:

$$\hat{\mathcal{O}}|\psi\rangle = \hat{\mathcal{O}}\cdot\left(\sum_{|\phi\rangle\in Supp(|\psi\rangle)}^{n} \alpha_\phi|\phi\rangle\right) = \sum_{|\phi\rangle\in Supp(|\psi\rangle)}^{n} \alpha_\phi\hat{\mathcal{O}}|\phi\rangle = \sum_{|\phi\rangle\in Supp(|\psi\rangle)}^{n} \alpha_\phi|\mathcal{O}(|\phi\rangle)\rangle$$

For some finished board $|\psi\rangle$.

This unitary will replace the role of the outcome-function in the previous section. Applying $\hat{\mathcal{O}}$ to a superposition will determine the win-state of each board and by its construction will allow for more interference. The changed gameplay will be like so:

1. Combine the unitaries/moves of both players, as described previously.
2. Apply $\hat{\mathcal{O}}$.
3. Measure the final superposition to get an outcome in $\{X, O, T\}$.

The adapted quantum circuit for this version of QT3 is illustrated in figure 10.

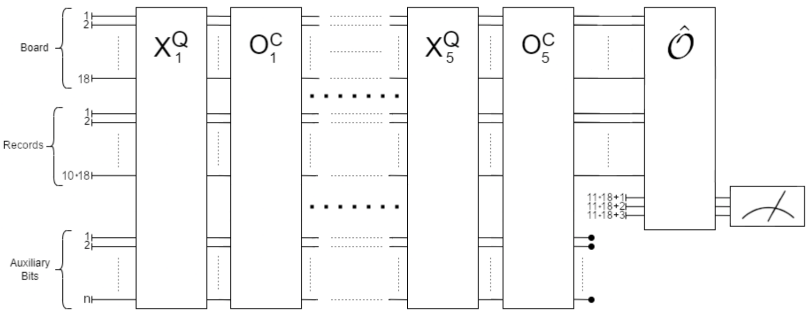


Figure 10: Quantum Tic-tac-toe circuit with delayed measurement

In the rest of this section, we describe how this change to QT3 does not affect the results of the previous section.

First, notice that when $|\psi\rangle \in |\mathcal{B}\rangle$, $\mathcal{O}(|\psi\rangle)$ coincides with $\hat{\mathcal{O}}|\psi\rangle$ after measurement. This means the definition of $B^{M,\vec{U}}_{M',\mathcal{M}}$ (def. 3.11) in the case of $\mathcal{O}$ coincides with swapping $\mathcal{O}$ with $\hat{\mathcal{O}}$ and measurement. This means the proofs of theorems 3.14 and the associated corollary 3.14.1.

The problem is with lemma 3.15. To prove the final result of the previous section, we first formulate the following alternative lemma.

Lemma 3.17. *Let $|\psi\rangle \in \langle\mathcal{B}\rangle$ a quantum board with 0 moves left to be made. If, writing $|\psi\rangle = \sum_{|\phi\rangle\in|\mathcal{B}\rangle} \alpha_\phi|\phi\rangle$, we have $\sum_{|\phi\rangle\in E} \alpha_\phi = 0$ where $E := \{|\phi\rangle \in Supp(|\psi\rangle) \mid \mathcal{O}(|\phi\rangle) = M\}$, then $\mathbb{P}(\text{player } M \text{ wins}) = 0$.*

Proof. Let $|\psi\rangle \in \langle\mathcal{B}\rangle$ be a quantum board with 0 moves left to be made, with $E := \{|\phi\rangle \in Supp(|\psi\rangle) \mid \mathcal{O}(|\phi\rangle) = M\}$.

The chance that the final superposition collapses to M can be expressed like so:

$$\begin{aligned}
\mathbb{P}(\text{player } M \text{ wins}) &= |\langle M\psi|\hat{\mathcal{O}}|\psi 0^3\rangle\rangle|^2 \\
&= |\langle M|\langle\psi|\psi\rangle \sum_{|\phi\rangle\in Supp(|\psi\rangle)} \alpha_{|\phi\rangle}|\mathcal{O}(|\phi\rangle)\rangle|^2 \\
&= |\sum_{|\phi\rangle\in Supp(|\psi\rangle)} \alpha_{|\phi\rangle} \cdot \langle M|\mathcal{O}(|\phi\rangle)\rangle|^2 \\
&= |\sum_{|\phi\rangle\in Supp(|\psi\rangle)} \alpha_{|\phi\rangle} \cdot \delta_{M,\mathcal{O}(|\phi\rangle)}|^2 = |\sum_{|\phi\rangle\in E} \alpha_{|\phi\rangle}|^2 = 0
\end{aligned}$$

□

Since the proof of corollary 3.14.1 still holds when measurement is postponed. We can now provide the following alternate proof of theorem 3.16 using lemma 3.17:

Proof. Let $|\psi\rangle$ be the quantum board after 10 moves, derived from the strategies chosen by M and $\overline{M}$. Corollary 3.14.1 gives:

$$Supp_{\{X,O,T\}}(\hat{\mathcal{O}}|\psi\rangle) \subseteq \mathcal{O}(Supp(|\psi\rangle)) \subseteq \{\overline{M}, T\}$$

Support regarding only the bits $18 \cdot 11 + 1,\ 18 \cdot 11 + 2,\ 18 \cdot 11 + 3$.

Let us assume that $\sum_{\phi\in E} \alpha_\phi \neq 0$, with E defined as in lemma 3.17. By the construction of E, we get:

$$\exists|\phi\rangle \in Supp(|\psi\rangle) \text{ with } \mathcal{O}(|\phi\rangle) = M \implies \{M\} \subseteq Supp_{\{X,O,T\}}(\hat{\mathcal{O}}|\phi\rangle)$$

Which implies that $M \in Supp_{\{X,O,T\}}(\hat{\mathcal{O}}|\psi\rangle)$ which gives a contradiction. By this contradiction, we may apply lemma 3.17 to conclude the proof.

□

With this revised proof of theorem 3.16 we show that the result of the previous section still holds for the case of postponed measurement.

3.6 Conclusion

In this chapter, we formally defined classical Tic-tac-toe and provided a novel quantization which is more general and less restrictive than existing quantizations. Furthermore, we discussed that the proposed Quantum Tic-tac-toe construction is a generalization of classical Tic-tac-toe. We proved that in this version of Quantum Tic-tac-toe the quantum player has no advantage over the classical player, the game will still always end in a tie — assuming both players play optimally. Finally, we discussed how a change in the definition of our quantization, namely changing the time of measurement, does not affect the results of this chapter.

In the next chapter, we show that if the opponent plays suboptimally then a quantum player *can* outperform an optimal classical player against the suboptimal classical player.

4 : Tic-Tac-Toe Against an Imperfect Opponent

In the previous chapter, we saw that the classical non-losing strategy is also non-losing against a quantum opponent. For that case, we assumed the opponent is a perfect player, but this is quite a strong assumption. In this section, we discuss what happens when the quantum player plays against a so-called "imperfect opponent".

We propose two different quantum strategies. We compare the performance of these quantum strategies against a classical imperfect opponent to the performance of a classical optimal strategy against the same classical imperfect opponent. We first describe what makes an opponent imperfect and what it means to make a mistake. Following that, we describe and simulate the quantum strategies and show how these measure up to a classical optimal strategy.

4.1 Imperfect Opponents

In this section, we consider the scenario where one player M plays against an imperfect opponent $\overline{M}$. Player $\overline{M}$ (the "imperfect opponent") plays a classical non-losing strategy as defined in 3.12 but with some *mistakes*. Player M either plays a classical optimal strategy or a certain quantum strategy. We experimentally determine the chance of player M winning in either case and try to find a quantum strategy for M which maximizes this chance and compare it to the classical strategy.

The probability of player M winning with a classical non-losing strategy depends highly on the strategy chosen. We can see this in the following example (where M plays as X):

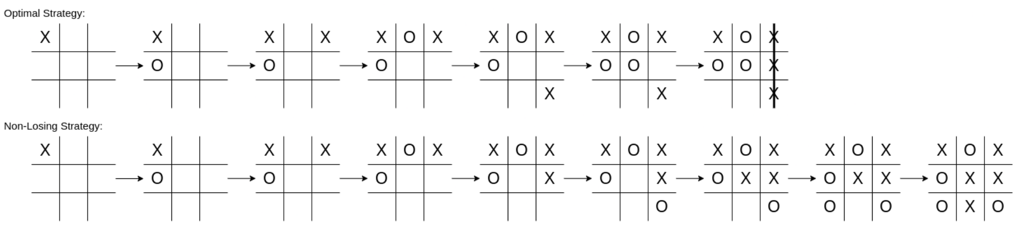


Figure 11: Optimal Strategy and Non-losing Strategy example

We can see that, in both cases, player X plays a non-losing strategy, but in the optimal case we see player X also *wins*. Note that this difference only occurs since player O made a mistake — this is why the difference between non-losing and optimal was not discussed in the previous chapter 3. To get useful results in the simulations we need to be more specific. This leads us to define an *optimal strategy*. First, we need to define when a board is winnable.

Definition 4.1. Let $\psi \in \mathcal{B}$ be a valid reachable board, with either $2n$ or $2n+1$ moves left to be made, $0 \leqslant n \leqslant 5$. We call ψ, M **winnable by force**, $M \in \{X, O\}$. If:

$$\exists U_n, \ldots, U_5 \in \mathbb{U}_C \ \forall V_{n-1}, \ldots, V_5 \in \mathbb{U}_C : \mathcal{O}(V_5 U_5 \ldots V_n U_n |\psi\rangle) = M$$

if M is to move on ψ, else:

$$\exists U_n, \ldots, U_5 \in \mathbb{U}_C \ \forall V_n, \ldots, V_5 \in \mathbb{U}_C : \mathcal{O}(U_5 V_5 \ldots U_n V_{n-1} |\psi\rangle) = M$$

Now we define the set $\mathcal{B}_M := \{\psi \in \mathcal{B} \mid \psi \text{ is } M \text{ winnable by force}\} \subseteq \mathcal{B}$. We define the set $\mathcal{B}_\mathcal{T} := \mathcal{B} \backslash (\mathcal{B}_X \cup \mathcal{B}_O)$.

We defined classical non-losing strategies in the previous section (definition 3.12). Take two classical non-losing strategies P and R and some $|\psi\rangle \in \mathcal{B}_X$. Now, P might force the win for X while R forces a tie. Both are still classical non-losing strategies, as described in the previous section and illustrated in figure 11, but we may consider P to be the superior strategy. We now define a classical optimal strategy, as the strategy which forces all boards which are M winnable by force to an actual win for M.

Definition 4.2. Let $\hat{P}$ be a classical non-losing strategy. $\hat{P}$ is called **optimal** if:
For all $|\psi\rangle \in \mathcal{B}_{\overline{M}}$,

$$\forall U_n, \ldots, U_5 \in \mathbb{U}_C : \mathcal{O}(\hat{P}U_5 \ldots \hat{P}U_n|\psi\rangle) = \overline{M}$$

if M is to move, else:

$$\forall U_m, \ldots, U_5 \in \mathbb{U}_C : \mathcal{O}(U_5\hat{P} \ldots U_m\hat{P}|\psi\rangle) = \overline{M}$$

Remark. Such an optimal strategy $\hat{P}$ exists, because we can construct such a strategy on a classical computer. This can be done by complete enumeration over all moves and possible continuations of the game (we do not care about computation time). This algorithm is also implemented as part of the simulations in the rest of this chapter (see GitHub) Note that this $\hat{P}$ is not unique and in the case that multiple options are "equally optimal" we choose one randomly.

Now we want to define what it means for $\overline{M}$ to make mistakes. We define mistakes based on $\mathcal{B}_M$ and $\mathcal{B}_T$ from definition 4.1.

Definition 4.3. Let $M \in \{X, O\}$. Let $\mathscr{E}_{\overline{M}} := \{\psi \in \mathcal{B}_\mathcal{T} \mid \exists U \in \mathbb{U}_C : U|\psi\rangle \in |\mathcal{B}_M\rangle\}$ be the set of boards upon which an effective mistake (error) can be made by $\overline{M}$. Let $U \in \mathbb{U}_C$ be a classical quantum move and $\psi \in \mathscr{E}_{\overline{M}}$, we then call player $\overline{M}$ applying U to $|\psi\rangle$ player $\overline{M}$ making a **mistake** if:

$$U|\psi\rangle \in |\mathcal{B}_M\rangle$$

Remark. Not every board is open for errors by $\overline{M}$. If a board is M winnable by force, $\psi \in \mathcal{B}_M$, then player $\overline{M}$ cannot make the situation any worse, and thus we consider it impossible to make a mistake. If there is no move U such that $U|\psi\rangle \in |\mathcal{B}_M\rangle$ for some $\psi \in \mathcal{B}_T$, then player $\overline{M}$ effectively cannot make a mistake on board $|\psi\rangle$. Lastly, we do not consider the case where $\psi \in \mathcal{B}_{\overline{M}}$, since both the classical and quantum strategies of player M, which we construct, will ensure no intermediate boards will be in $\mathcal{B}_{\overline{M}}$. This can be achieved by using some form of the *non-losing strategy*, of which an optimal strategy is a special case.

4.2 Simulation of Quantum Strategies

To study the potential advantage quantum strategies may provide we will perform a number of simulations, and we will show that we can design a quantum strategy which provides us with a significantly better chance of winning compared to a classical optimal player.

The simulations are run in Python, and the code can be found on GitHub. Before the games are "played" all possible boards are categorized in the categories, $\mathcal{B}_X$, $\mathcal{B}_O$, and $\mathcal{B}_T$. This is done inductively by starting at all boards which are filled completely, excluding any which are invalid (so a board where both players could be deemed winner). These boards were generated by simply regarding all permutations of five X's and four O's, we did not implement any efficiency measures regarding rotations and reflections equivalency. This was done since the current speed of the program is manageable and correctness is easier to verify in this case.

We then progressively removed either an X or O from consideration and looked at all permutations again. For instance, in the second iteration, we consider all permutations of four X's and four O's, again disregarding any invalid boards. We classify these boards based on the classification of the boards they can move to, or — if finished — we simply determine the winner and classify accordingly.

Finally, for boards in $\mathcal{B}_T$ we also check whether player M, the player-to-move on the board, can move to $\mathcal{B}_{\overline{M}}$. This gives us $\mathscr{E}_X$ and $\mathscr{E}_O$, stored as one dictionary in the code since the player can be inferred.

We ensure we only regard valid reachable boards by (i) checking the number of winners, (ii) the number of X's and O's (in general the difference may not be greater than one), and (iii) that players did not make one additional move after a player wins (this is not always caught by checking if the difference is $\leqslant 1$).

Games are played by applying two strategies sequentially on the initial board $|0\rangle$ and observing the outcomes. The boards played on are quantum boards, which is simply a set of classical boards with associated amplitudes. We store the absolute squares of the amplitudes. This is done to reduce numerical errors. In addition, it does not affect simulations, since we choose the quantum strategies in such a way that we do not encounter any complex amplitudes.

We define different strategies as described in this chapter, where classical strategies are applied on each board in the support of the quantum board, and their results are aggregated if needed.

Measurement is performed by uniformly generating a number $r \in [0, 1]$. Given some ordering of the boards; the superposition collapses on board i if

$$|\alpha_0|^2 + \cdots + |\alpha_{i-1}|^2 < r \leqslant |\alpha_0|^2 + \cdots + |\alpha_{i-1}|^2 + |\alpha_i|^2$$

When simulation a strategy we run multiple batches of games. The average number of tied games and the standard deviation is determined between these batches, which is plotted with error regions (error regions are the size of two standard deviations). The winner is determined as it is classically defined.

For each simulation, we simulate three different strategies: am imperfect strategy ($\overline{M}$), a classical optimal strategy (M^C), and a quantum strategy (M^Q). The classical optimal strategy will choose uniformly from all optimal (next) moves (i.e. moves that stay in $\mathcal{B}_M$), as defined in definition 4.2. The imperfect strategy will act the same as the classical optimal strategy, except for that it makes a mistake on boards in $E \subseteq \mathscr{E}_{\overline{M}}$, where a board in $\mathscr{E}_{\overline{M}}$ has chance p to be in E. We can denote the effects of each move by:

$$\overline{M}(|\psi\rangle) = \begin{cases} |\phi\rangle, \psi \xrightarrow{\gamma} \phi \in \mathcal{B}_M, & \text{if } \psi \in E \\ |\phi\rangle, \psi \xrightarrow{\gamma} \phi \in \mathcal{B}_T, & \text{if } \psi \in \mathcal{B}_T \end{cases}$$

With γ a valid classical move for player $\overline{M}$.

$$M^C(|\psi\rangle) = \begin{cases} |\phi\rangle, \psi \xrightarrow{\lambda} \phi \in \mathcal{B}_M, & \text{if } \psi \in \mathcal{B}_M \\ |\phi\rangle, \psi \xrightarrow{\lambda} \phi \in \mathcal{B}_T, & \text{if } \psi \in \mathcal{B}_T \end{cases}$$

With λ a valid classical move for player M.

Remark. These "definitions" are ambiguous and incomplete. First, we address the incompleteness, we have not specified what happens for $\overline{M}(|\psi\rangle)$ when $\psi \in \mathcal{B}_{\overline{M}}$, in the case of this simulation this is not important since this case cannot occur. Both strategies,

M^C and M^Q, are defined/will be defined non-losing, so after a move made by either strategy the resulting position cannot be in $\mathcal{B}_{\overline{M}}$. In addition, we stipulate that the definitions for $\overline{M}$ and M^C are given for boards in $|\mathcal{B}\rangle$, by linearity this can be extended to all positions in $\langle\mathcal{B}\rangle$.

Secondly, ambiguity arises when there are multiply optimal moves. In this case, like explained before, we choose uniformly from all possible optimal next moves and the decision is stored to ensure the strategy stays deterministic (in theory not in the actual implementation).

Finally, in the coming sections, we will describe three different quantum strategies M^Q. For each quantum strategy we ran 5 simulations or 350 games with one player playing a classical imperfect strategy and the other player playing either a classical optimal strategy or the quantum strategy — these strategies will be described in more detail below 4.3, 4.4. The simulations were run for p — the probability of making mistakes — from 0 to 1 in increments of $\delta = 0.01$. In addition, simulations were run for both the imperfect player as player X and as player O. Finally, the results will be plotted in two graphs.

4.3 Simple Quantum Strategy

We first perform a simple simulation where we only exploit the ability to create superpositions. At every move we play, we make an equally distributed superposition of optimal moves. We can express this move, M^Q like so:

$$
\begin{aligned}
M^Q|\psi\rangle = \sum_{\phi\in Supp(|\psi\rangle)} & \frac{1}{\sqrt{\max(1,\ |\{\kappa\in\mathcal{B}_M|\phi \xrightarrow{\gamma} \kappa\}|)}} \sum_{\phi\xrightarrow{\gamma}\kappa\in\mathcal{B}_M} |\kappa\rangle\cdot\mathcal{X}\{\phi\in\mathcal{B}_M\} \\
& + \frac{1}{\sqrt{\max(1,\ |\{\kappa\in\mathcal{B}_T|\phi \xrightarrow{\gamma} \kappa\}|)}} \sum_{\phi\xrightarrow{\gamma}\kappa\in\mathcal{B}_T} |\kappa\rangle\cdot\mathcal{X}\{\phi\in\mathcal{B}_T\}
\end{aligned}
$$

Remark. M^Q is non-losing.

Remark. We add the $\max(1,\dots)$ to ensure that the amplitude is well-defined, even if there are no valid moves. Notice that when $\max(1,\dots)$ "chooses" 1, the sum associated with that term will evaluate to 0.

We call the "game tree", the tree with root the empty board, and with each node in the tree representing a board ψ. Its children $\phi_1,\dots,\phi_n$ are boards where $\psi \xrightarrow{[n_M]} \phi$ is a valid move for some field marking $[n_M]$. If the game tree would be "perfect" — i.e. each node in a layer would have the same number of children and, if we look at the tree as an undirected graph, it does not contain cycles — this Simple Quantum Strategy should not help. This can be seen by noticing that each classical game played is a path through the tree and it does not matter if you choose a path by classically choosing moves or making equally distributed superpositions of moves. In both cases, collapsing the final superposition will yield final boards with same probability. We illustrate this concept in a simplified tree figure.

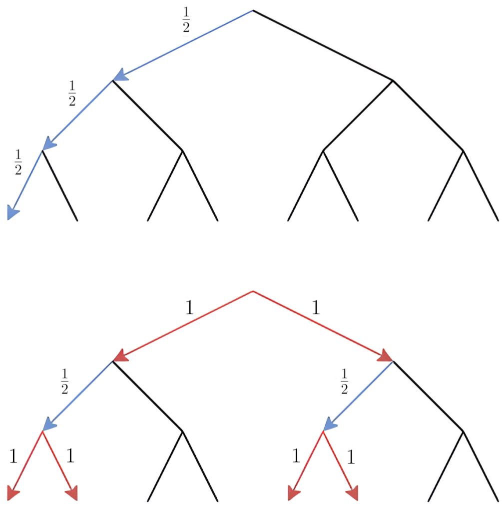


Figure 12: Simple Quantum Strategy (red) on a perfect tree, and classical strategy (blue) on the same tree.

In this overly simplified example, we can see that the probability of getting to the left-most leaf in the first tree (only classical moves, where each split has a $\frac{1}{2}$ chance of going left or right) is $\frac{1}{8}$. In the second tree, the probability of the left-most leaf being in the final superposition is $\frac{1}{2}$. Since each leaf in this superposition has the same amplitude of $\frac{1}{\sqrt{4}}$ we get that the chance of measuring the left-most leaf is $\frac{1}{4}$. This means that the chance of having the left-most leaf be the final result in the lower tree also has probability $\frac{1}{8}$. This is of course a simplified example, but it illustrates the line of reasoning.

In contrast to this example, the actual game will have an imperfect game tree, so it might still be possible that some advantage can be gained.

We ran the simulations, and we got the following results. Here we plot the number of *ties* against the probability of the imperfect player making a mistake p.

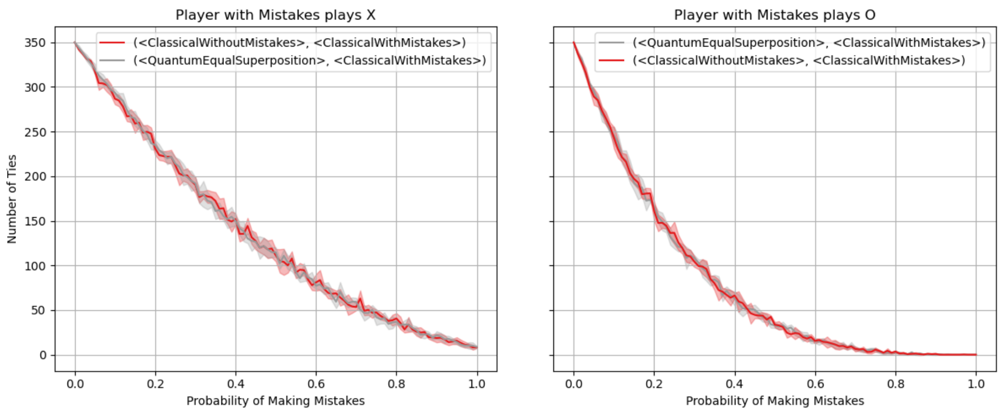


Figure 13: Plot of performance Simple Quantum Strategy and Classical Strategy

Remark. The number of ties is plotted. By construction of the strategies we cannot lose. This means: # wins = 350 − # ties, so fewer ties means more wins.

We see no significant difference between the two players' performance. That is, the red and grey lines and regions coincide. What is notable is that when the imperfect player plays as X the graph does not reach zero ties for a probability of making a mistake of one. This is because if the imperfect opponent plays as X it is possible to "accidentally" choose a path upon which it encounters no boards where mistakes are possible (see figure 14). This is not the case for when the imperfect opponent plays as O, since on each board directly after the starting board a mistake can be made.

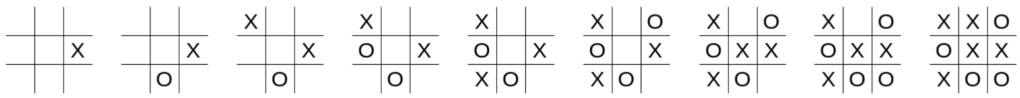


Figure 14: Game where player X can not make mistakes

We also see that in general the performance is better if the imperfect opponent plays as player O. The can explained by the fact that there are more boards where player O can make mistakes on, than boards where player X can make mistakes on. Since O has more opportunities to make mistakes, naturally, the imperfect opponent will perform worse as O.

From the results, it seems that this simple quantum strategy will still not outperform a classical optimal strategy. Next, we will introduce a different quantum strategy not relying on just superpositions. This new strategy will make use of "Amplitude Amplification" to gain an advantage over a classical player playing a classical optimal strategy, when playing against an imperfect classical player.

4.4 Amplitude Amplification Strategy

This strategy is an expansion on the previous strategy. We will use *Amplitude Amplification* to "amplify" winning boards, making them more likely to be measured. Amplitude amplification (as described in [1]) is similar to Grover's Search [9] but applicable in more general situations, which is required in our case. This strategy, can be seen as performing the same strategy as before, the "Simple Quantum Strategy", but at the end it "searches" through all the positions to find winning positions. Like in Grover's Search the "searching" will be done by amplifying the states (boards) which meet the searching criteria. We will show — by simulations — that this strategy is significantly better than a classical strategy.

We will first explain all the applicable theory behind Amplitude Amplification as defined in [1].

4.4.1 Amplitude Amplification

First, we need an oracle $\mathcal{O}_f$ which can distinguish winning and non-winning boards. $\mathcal{O}_f$ will be defined like so

$$\mathcal{O}_f|\psi R\rangle = (-1)^{f(|\psi R\rangle)}|\psi\rangle$$

where

$$f(|\psi R\rangle) \begin{cases} 1, & \text{if } \psi \in \mathcal{B}_M \\ 0, & \text{if } \psi \notin \mathcal{B}_M \end{cases}$$

for $|\psi R\rangle \in |\mathcal{B}\rangle$ and some player $M \in \{X, O\}$. Since we can express all classical algorithms as quantum algorithms, and since we can express f as a classical algorithm (see [GitHub]), we can assume this $\mathcal{O}_f$ exists.

The second part of this strategy is a unitary $\mathcal{A}$ with the following property:

$$\mathcal{A}|0\rangle = |\psi\rangle$$

Where $|\psi\rangle$ is the state of the game to which we will be applying amplitude amplification. This comes with a problem, namely, to construct $\mathcal{A}$ we need the moves of the opponent. In the current definition of the game we did not specifically specify if this is allowed or not. So we further specify gameplay by saying that each player applies their unitary (their move) to the game and then shows the unitary to the other player. Note that this does not reveal any information because the players are free to change their unitary each move they make, so the player does not need to lay out their entire strategy at move one. Alternatively, we could say that as the quantum player you are also the operator of the quantum computer, and thus you can have a quick peek at the opponents unitaries, or something like that. In section 4.4.5, we discuss why an effective "blind" strategy — a strategy without using the opponents moves — is so hard to make. For the rest of this section, we assume we have some $\mathcal{A}$ with the property given above for the final position the quantum player moves on in the game, which will from now on be denoted by $|\psi\rangle$.

We introduce some new notation and terms to keep inline with the paper [1]. All following theory of this section 4.4.1 comes from [1]. We call $\mathcal{H} = \langle\mathcal{B}\rangle$ the *search space*. We also need to split up this space in *good states* and *bad states*, in our case, these are winning and non-wining boards for player M respectively. The *good states* are defined as

$$\mathcal{H}_1 := \text{Span}_{\mathbb{C}}(|\mathcal{B}_M\rangle)$$

and the *bad states* are defined as

$$\mathcal{H}_0 := \mathcal{H}_1^{\perp}$$

With this division of the search space into good and bad states we can uniquely decompose $|\psi\rangle$ as

$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$$

where $|\psi_0\rangle \in \mathcal{H}_0$ and $|\psi_1\rangle \in \mathcal{H}_1$ are orthogonal but not normal. The final goal is to get a new final board where the chance of collapsing to a position in $Supp(|\psi_1\rangle)$ is significantly greater than it is for $|\psi\rangle$.

The current situation are two vectors in two orthogonal hyperplanes, $\mathcal{H}_0$ and $\mathcal{H}_1$. One can picture:

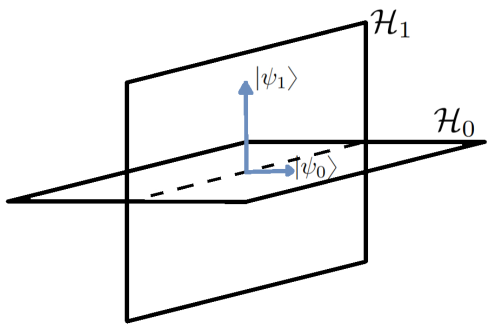


Figure 15: Space spanned by $\mathcal{H}_0$ and $\mathcal{H}_1$

For amplitude amplification we will only be considering the plane spanned by the vectors $|\psi_0\rangle$ and $|\psi_1\rangle$.

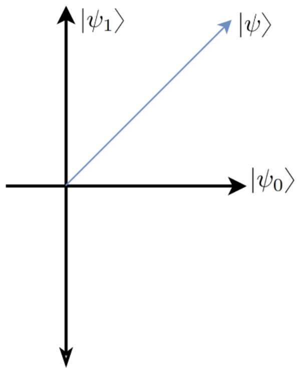


Figure 16: Starting state of amplitude amplification.

Notice that $|\psi\rangle$ has an angle of 45 degrees to $|\psi_0\rangle$ and $|\psi_1\rangle$. In Grover's Search you, generally, do not see an initial angle of 45 degrees. In most cases, the angle is often "skewed" to either axis. The illustration above scales the vectors $|\psi_0\rangle$ and $|\psi_1\rangle$ to the same length, which shows a angle 45 degrees. The skewness we expect is hidden in the amplitudes of $|\psi_0\rangle$ and $|\psi_1\rangle$, which can be seen in figure 15 where $|\psi_0\rangle$ and $|\psi_1\rangle$ have different lengths.

We now apply a unitary Q, which will perform the amplification of positions in $Supp(|\psi_1\rangle)$ — in the case of Grover's Search this is called the *Grover Operator*. We will refer to Q as the *Amplifier*. We define Q as

$$Q := -\mathcal{A}O_{\chi_{\{0\}}}\mathcal{A}^\dagger O_f$$

Where $O_{\chi_{\{0\}}}$ is the reflection across $|0\rangle$. To best explain the working of Q we split it up into O_f, a reflection across $|\psi_0\rangle$, and $-\mathcal{A}O_{\chi_{\{0\}}}\mathcal{A}^\dagger$, a reflection across $|\psi\rangle$. We can illustrate O_f like so:

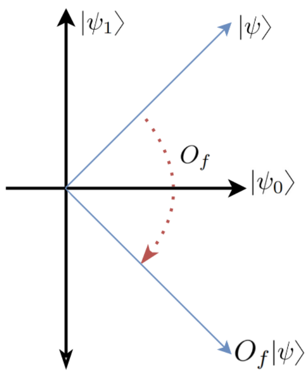


Figure 17: Effects of O_f.

We just mentioned $-\mathcal{A}O_{\chi_{\{0\}}}\mathcal{A}^\dagger$ is a reflection across $|\psi\rangle$: this is nearly correct. The actual operation is a reflection across $|\psi\rangle$ times -1. This can be seen with a simple calculation.

$$-\mathcal{A}O_{\chi_{\{0\}}}\mathcal{A}^\dagger = -\mathcal{A}(I - 2|0\rangle\langle 0|)\mathcal{A}^\dagger = 2|\psi\rangle\langle\psi| - I$$

We can illustrate this geometrically.

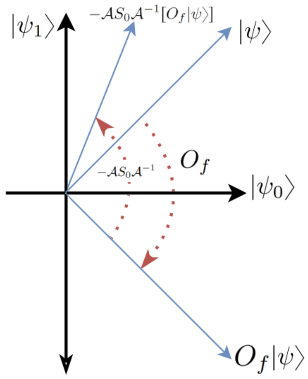


Figure 18: A graphical illustration of the effects of $-\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger$.

Remark. The illustration does not show length preservation, however this is a consequence of axis rescaling (see fig. 15).

We also want to explicitly calculate the effect Q has on $|\psi\rangle$. To do this we provide some intermediate calculations. First, we calculate the effect of $-\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger$ on $|\psi_0\rangle$ and $|\psi_1\rangle$. Let $a = \langle\psi_1|\psi_1\rangle$. We assume $0 < a < 1$, since in the case that $a = 0$ or $a = 1$ the amplifier Q has no effect; it only affects the global phase.

$$\begin{aligned}
-\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger|\psi_0\rangle &= (2|\psi\rangle\langle\psi| - I)|\psi_0\rangle \\
&= 2[|\psi_0\rangle + |\psi_1\rangle][\langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_0\rangle] - |\psi_0\rangle \\
&= 2[|\psi_0\rangle + |\psi_1\rangle](1-a) - |\psi_0\rangle \\
&= (1-2a)|\psi_0\rangle + 2(1-a)|\psi_1\rangle
\end{aligned}$$

$$\begin{aligned}
-\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger|\psi_1\rangle &= (2|\psi\rangle\langle\psi| - I)|\psi_1\rangle \\
&= 2[|\psi_0\rangle + |\psi_1\rangle][\langle\psi_0|\psi_1\rangle + \langle\psi_1|\psi_1\rangle] - |\psi_1\rangle \\
&= 2[|\psi_0\rangle + |\psi_1\rangle]a - |\psi_1\rangle \\
&= 2a|\psi_0\rangle + (2a-1)|\psi_1\rangle
\end{aligned}$$

Note that these results are not normalized, since $|\psi_0\rangle$ and $|\psi_1\rangle$ are not either. We can now apply Q to $(\alpha|\psi_0\rangle + \beta|\psi_1\rangle)$, this gives:

$$\begin{aligned}
Q(\alpha|\psi_0\rangle + \beta|\psi_1\rangle) &= -\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger[O_f(\alpha|\psi_0\rangle + \beta|\psi_1\rangle)] \\
&= -\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger[\alpha|\psi_0\rangle - \beta|\psi_1\rangle] \\
&= -\alpha\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger|\psi_0\rangle + \beta\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger|\psi_1\rangle \\
&= (\alpha(1-2a) - 2\beta a)|\psi_0\rangle + (2\alpha(1-a) - \beta(2a-1))|\psi_1\rangle
\end{aligned}$$

Note that assuming $(\alpha|\psi_0\rangle + \beta|\psi_1\rangle)$ is normalized than this result is also normalized. This can be seen by noticing that Q is a product of unitaries. We can also explicitly calculate this, we show it here for $|\psi\rangle$:

$$\begin{aligned}\|Q|\psi\rangle\| =& \| -\mathcal{A}O_{\mathcal{X}_{\{0\}}}\mathcal{A}^\dagger[O_f|\psi\rangle]\| \\ =&\|(1-4a)|\psi_0\rangle + (3-4a)|\psi_1\rangle\| \\ =&(1-4a)^2\langle\psi_0|\psi_0\rangle + (3-4a)^2\langle\psi_1|\psi_1\rangle \\ =&(16a^2-8a+1)(1-a) + (16a^2-24a+9)(a) \\ =&16a^2-8a+1-16a^3+8a^2-a+16a^3-24a^2+9a \\ =&1\end{aligned}$$

Just like with Grover's Search we do not apply Q once but, we apply Q multiple times to get an even better result. Let θ be such that:

$$\sin^2\theta = a$$

j applications of Q on $|\psi\rangle$ gives:

$$Q^j|\psi\rangle = \frac{1}{\sqrt{a}} \cdot sin((2j+1)\theta) \cdot |\psi_1\rangle + \frac{1}{\sqrt{1-a}} \cdot cos((2j+1)\theta) \cdot |\psi_0\rangle$$

For brevity, we do not give the derivation for this result (this is worked out in [1]). We now want to find j such that the amplitude of $|\psi_1\rangle$ is maximal. To do this it is clear that we want $(2j+1)\theta \approx 2\pi$, which gives that the amplitude of $|\psi_1\rangle$ is maximal for $j = \left[\frac{\pi}{4\theta} - \frac{1}{2}\right] = \left\lfloor\frac{\pi}{4\theta}\right\rfloor$; we denote this optimal j by n.

So for applying Q^n provides optimal amplification of the good states. We now encounter the problem of needing to know n. We remark that this is not as simple as the above derivation for n, since we do not have θ, as amplitudes (and therefore a) are in principle unknown when running on a quantum computer[6]. Of course, we could determine a by using *Phase Estimation* [9], which would work well enough since we are not concerned with computation time and thus can assume a negligible error in the approximation of a, however in the current setup of the game we have $\mathcal{A}$ explicitly and can use it to simply calculate the amplitudes. In any case, for the rest of this section we can assume we compute n correctly.

4.4.2 Difference with Grover's Search

We will briefly explain why we do not simply do Grover's Search, since it is so similar. The main difference is that Grover's Search is designed to search through $\{0,1\}^m$ (for some integer m), while Amplitude amplification allows searching through a specified subset of this space — like all boards in the current superposition. Grover's Search does not use the amplifier Q, but the Grover Operator G:

$$G = (2|\phi\rangle\langle\phi| - I)O_f$$

Conventionally $|\phi\rangle = H^{\otimes n}|0^n\rangle$, an equally distributed superposition over the entire search space. The problem with this is that in our application we want to search through a subset of the search space, namely the support of $|\psi\rangle$. If we applied a Grover Iteration, we would introduce new boards into the superposition, which would not guarantee a valid move. Therefore, applying Grover's Search would imply making an illegal move because the chance of measuring any winning board increases to a non-zero chance.

[6] As mentioned in section 4.2, when simulating quantum computers on a classical computer amplitudes *are* known.

4.4.3 Simulations

The simulation of this strategy is very similar to the previous strategy (the *Simple Quantum Strategy* section 4.3), which only made equal superpositions of all good moves. For the first four moves of the quantum player we apply the exact same strategy. For the fifth move we again play a move which performs an equally distributed superposition of all optimal moves, but now we follow this move with amplitude amplification. Since we assume we can approximate n sufficiently well to deem errors negligible, in the simulations we determine n following the exact formula (which is possible when simulating on a classical computer). The results are plotted like the results of the previous strategy.

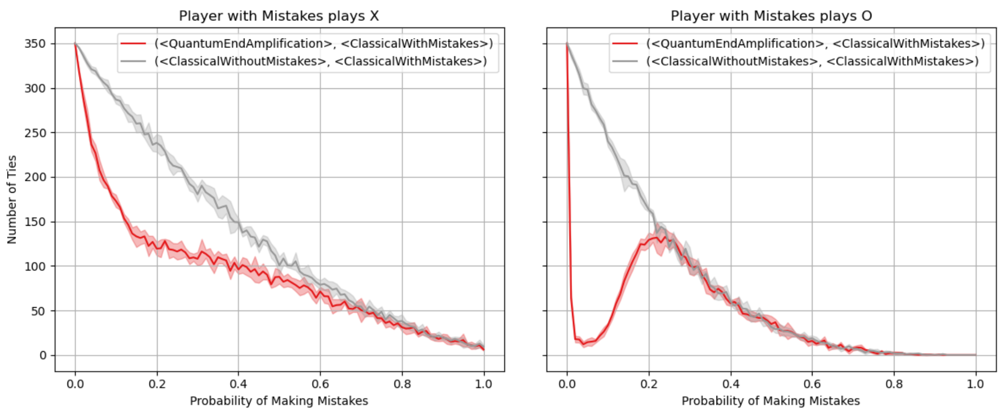


Figure 19: Performance Amplitude Amplification (at the last move) and Classical Strategy

First, we see a clear improvement over the classical strategy and the simple quantum strategy. We also see that this advantage wears off when the chance of making a mistake increases, which is expected since for high p there is little advantage to be gained.

There is an increase in the number of ties when the imperfect player plays as O at a mistake probability of about $p = 0.1$. We speculate this is because Amplitude amplification is most effective when the amplitude of the good states is low, and it is especially bad if the amplitudes of the good and bad states are very close to each other. We can see that at around $p = 0.2$ half of the games end in a tie, since the amplitudes are close Amplitude amplification is not as effective. For $p > 0.2$ the amplitudes of good states are higher than those bad states, and we then also see that Amplitude amplification is not as effective.

We have not proven this, however we presume that this steep increase is not present when the imperfect opponent player plays as X simply because it is less likely to make a mistake as player X. For example, on the first move as player X it is not possible to make a mistake, while on the first move as player O on each board you may get you may make a mistake.

We speculate the chance of winning, when p is small, as player X is too low to amplify effectively. We also see this when the opponent, player O, as for very low p the chance of ties is also very high. However, if the opponent plays as O with a slightly higher p, probability of making mistakes, the chance of winning as the quantum player increase to such a height that amplification is very effective, we estimate this value to be around $p = 0.1$.

We will test this hypothesis, however, by formulating a new strategy, which does amplification after *every* move it makes. Since chances of making mistakes increase when making more moves we would like to "catch" the mistakes before too many are made, so collapsing to winning boards gets too likely. We do this by amplifying earlier, or in the case of this strategy every time. The simulations give the following results, again plotted as the results of the previous two proposed strategies.

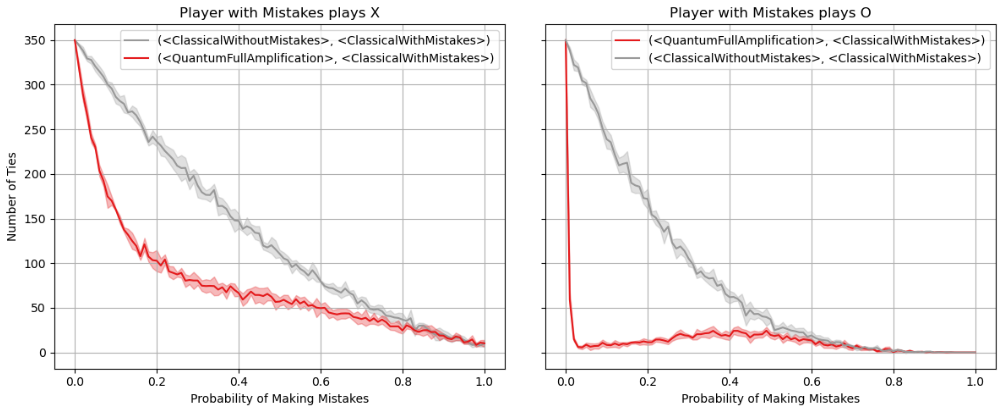


Figure 20: Performance Amplitude Amplification (at every move) and Classical Strategy

We see significant improvement over amplifying once. As stated before, we hypothesize this is due to the probabilities of making mistakes being just small enough to benefit from amplification the most. We believe the disappearance of the steep increase from $p = 0.1$ to $p = 0.2$ supports this claim, although we still see a slight bump. We leave further investigating this to future research. What we can see from the results are three things:

- When a classical opponent plays imperfectly the quantum player has a significant advantage over an optimal classical player.
- The quantum player using the amplitude amplification strategy should prefer playing as player X.
- The quantum player would prefer playing against an imperfect player who almost never makes mistakes opposed to an imperfect player who makes mistakes about half of the time.

Finally, we also provide a plot of the performance of all strategies together.

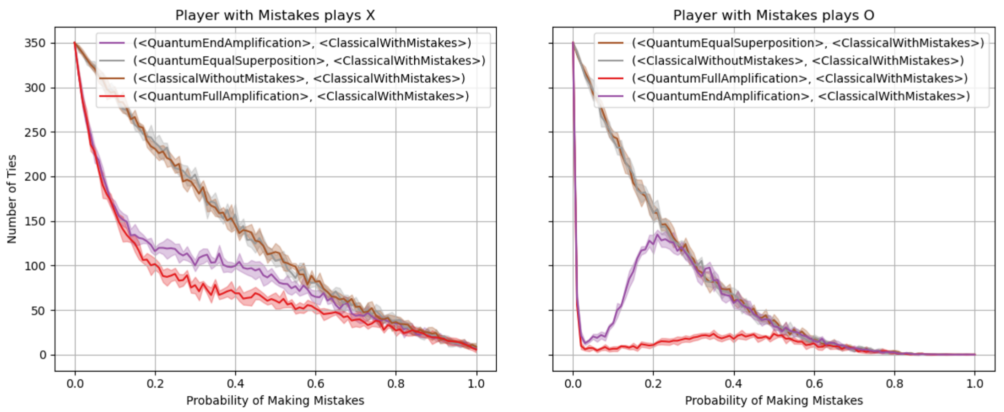


Figure 21: Performance of all strategies

4.4.4 Validity of Searching Moves

A "searching move", like the moves described above, moves where we search through the superposition for good boards are valid moves. This can be easily seen since they do not introduce any new positions. We can prove that applying the amplifier Q after a valid move still is a valid move. We first assume that the strategy proposed in Simple Quantum strategy 4.3 is made up of valid moves, which is true by definition of the strategy.

Lemma 4.4. *Let U be some valid quantum move such that $U|\phi\rangle = |\psi\rangle$ for some $|\psi\rangle, |\phi\rangle \in \langle\mathcal{B}\rangle$. Let Q be an amplifier for the position $|\psi\rangle$. Let $\mathcal{A}|0\rangle = U|\phi\rangle = |\psi\rangle$. Then Q^nU is a valid quantum move, for all $n \in \mathbb{N}$.*

Proof. First, note that Q and U are unitaries, therefore there products is too.

Take some $|\kappa\rangle \in Supp(QU|\phi\rangle)$:

$$\begin{aligned}|\kappa\rangle \in\ & Supp(QU|\phi\rangle)\\ &= Supp(Q|\psi\rangle)\\ &= Supp((1-4a)|\psi_0\rangle + (3-4a)|\psi_1\rangle), \quad a = \langle\psi_1|\psi_1\rangle\\ &\subseteq Supp(|\psi\rangle)\end{aligned}$$

Since U is a valid move we get that there must exist some position $|\kappa'\rangle \in Supp(|\phi\rangle)$ such that

$$|\kappa'\rangle \xrightarrow{\gamma} |\kappa\rangle$$

is a valid move for player $M \in \{X, O\}$. This gives that QU is a valid quantum move. Since QU is a valid quantum move we get that Q^nU is valid quantum move by applying the above reasoning n times. □

4.4.5 Blind Strategies

We can imagine that the Amplitude Amplification technique we proposed can be seen as a bit of a "cheaty" method or at least as not the most general strategy, considering that we must adapt the game such that we allow for non-blind strategies. A more general or fairer approach would be to not allow the quantum player to let their strategy depend on the moves of the opponent. Both players would determine their strategies before the game starts.

Nevertheless, the strategy we proposed, using Amplitude Amplification, only requires knowledge of prior moves as a "black box". So we still say that it is reasonable to regard the technique as a serious strategy. However, it is still worth discussing if there are no alternatives.

As we have discussed section 4.4.2, it is not possible to do Grover-Search-like moves without having a unitary $\mathcal{A}|0\rangle = |\psi\rangle$. The problem is that we need to find some method of increasing amplitudes without introducing new boards. We could look at something like Quantum Random Walks, but here we would need some oracle which could determine if a state is in the superposition or not. Such an oracle is possible, but the "output qubits" would be in superposition with the superposition $|\psi\rangle$, which is not sufficient to ensure valid moves.

This reinforces our view that the Amplitude Amplification strategy is a reasonable strategy, since it is the only one we know of which we can expect to deliver a significant advantage over a classical player. We leave finding a more general strategy to future research.

4.5 Conclusion

In this chapter, we defined what it means for a player to be imperfect. We used this imperfect player to show that quantum players *do* have an advantage over classical players when playing against a classical imperfect player. We did this through simulations, for which the code is available on GitHub [GitHub] To show this advantage we proposed a strategy based on *Amplitude Amplification*, we explained how this technique works and showed that it performs significantly better, especially for almost perfect imperfect players.

Conclusion and Discussion

In this thesis, we have proven that — in our definition of Quantum Tic-tac-toe — quantum strategies have no advantage over their classical counterparts. Furthermore, while quantum players cannot outperform classical players playing optimally, we observed that when the classical opponent makes mistakes, a quantum player can exploit this more effectively than classical players.

These results suggest that — beyond the well-known computational speed-ups — quantum computing does not provide a strategic advantage over classical computers in such a structured game as Tic-tac-toe. While classical strategies stay competitive in these settings, in more realistic scenarios where players are fallible, quantum strategies might give a significant edge. This may indicate that quantum computers are better suited for environments involving uncertainty and errors — contexts more typical of real-life applications.

Although this thesis focused on a very general form of Quantum Tic-tac-toe, the quantization of the game was defined with making classical strategies possible to be played in the Quantum game. This was done to be able to see whether quantum strategies can outperform classical strategies. However, we may also relax this requirement and allow for "non-injective playable tactics". This would mean we remove the records from the positions. This would make playing classical strategies impossible, but it would open up the question of what quantum strategies are optimal in this setting.

The framework of quantizing Tic-tac-toe introduced in this thesis could be extended to other combinatorial games. When using the same quantization framework to quantize combinatorial games in general, future research could explore whether the conclusions of chapter 3 hold for all combinatorial games.

Finally, the simulation results from quantum Tic-tac-toe against imperfect opponents indicate that quantum strategies may offer practical advantages in non-ideal scenarios. It would be worthwhile to prove these findings for Tic-tac-toe. Future research could even generalize them to the broader class of combinatorial games, as described above. Moreover, future research could also explore blind strategies, which do not rely on the knowledge of prior strategies, and their potential advantages against imperfect opponents.

In conclusion, even though quantum computers provide many advantages over classical computers in various computational domains, our findings show that quantum computers cannot outperform classical computers in the context of Tic-tac-toe. This would suggest that even a human player can perform just as well as a quantum player, however, they may make no mistakes — otherwise, they are likely to lose. So, whenever it becomes possible to play games on a quantum computer, we need not worry — we just need to play non-losing.

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