Complexity exercise set #2

for the tutorial on
April 21, 2022

Exercises may be handed in for grading and can earn you a small bonus[1] on the exam, provided you submit your solutions via Brightspace in PDF before **15:15 on Monday April 25**.

Exercise 1 (20 points) Let $x \in \mathbb{R}$ and $n \in \mathbb{N}\backslash\{0\}$ be given. Prove in detail that

$$\lceil x/n \rceil \;=\; \lceil \lceil x \rceil / n \rceil . \tag{1}$$

You may use the fact that $\lceil y \rceil \leq m \iff y \leq m$ for all $y \in \mathbb{R}$ and $m \in \mathbb{Z}$.

Hint: first prove that $\lceil x/n \rceil \leq m \iff \lceil \lceil x \rceil / n \rceil \leq m$ for all $m \in \mathbb{Z}$.

Does equation (1) also hold for all $n \in (0, \infty)$ and $x \in \mathbb{R}$? If so, provide a proof; if not, give a counterexample, and indicate which step of your proof for the case $n \in \mathbb{N}\backslash\{0\}$ breaks down when $n \in (0, \infty)$.

Solution. Let $x \in \mathbb{R}$ and $n \in \{1, 2, \dots\}$ be given. Following the hint, we have, for any $m \in \mathbb{Z}$,

$$\begin{array}{rrcll}
 & \lceil \lceil x \rceil / n \rceil & \leq & m & \\
\iff & \lceil x \rceil / n & \leq & m & \text{by the given property of } \lceil - \rceil \\
\iff & \lceil x \rceil & \leq & mn & \text{because } y \mapsto yn \colon \mathbb{R} \to \mathbb{R} \text{ is monotone} \\
\iff & x & \leq & mn & \text{because } mn \text{ is whole} \\
\iff & x/n & \leq & m & \text{because } n \neq 0\text{, and } y \mapsto y/n \colon \mathbb{R} \to \mathbb{R} \text{ is monotone} \\
\iff & \lceil x/n \rceil & \leq & m & \text{by the given property of } \lceil - \rceil.
\end{array}$$

Taking $m = \lceil \lceil x \rceil / n \rceil$, we $\lceil \lceil x \rceil / n \rceil \leq m$, and so $\lceil x/n \rceil \leq m = \lceil \lceil x \rceil / n \rceil$. Similarly, taking $m = \lceil x/n \rceil$, we have $\lceil x/n \rceil \leq m$, and so $\lceil \lceil x \rceil / n \rceil \leq m = \lceil x/n \rceil$. Hence $\lceil \lceil x \rceil / n \rceil = \lceil x/n \rceil$.

Any $x \in \mathbb{R}$ and $n \in (0, \infty)$ with $x < n < \lceil x \rceil$ (such as $x = \frac{1}{2}$ and $n = \frac{2}{3}$) will furnish a counterexample (but there might be more.) To see why equation (1) fails for such n and x, note that $x < n < \lceil x \rceil$ implies that $x/n < 1 < \lceil x \rceil / n \leq \lceil \lceil x \rceil / n \rceil$, and so $\lceil x/n \rceil \leq 1 < \lceil \lceil x \rceil / n \rceil$.

The proof given above for when $n \in \mathbb{N}\backslash\{0\}$ breaks down at the step "$\lceil x \rceil \leq mn \iff x \leq mn$, because mn is whole," because mn need not be whole. ■

[1] For more details, see `https://cs.ru.nl/~awesterb/teaching/2022/complexity.html`.

Grading. **10 points** for a correct and detailed proof; **5 points** for any correct counterexample; **5 points** for pointing out at least one step that breaks down in *their* proof when $n \in (0, \infty)$.

Exercise 2 **(20 points)** Let $a \in (0, \infty)$ and $b \in (1, \infty)$ be given.

Let us call a function $f \colon \mathbb{N} \to [0, \infty)$ **regular**, when there are $c \in [0, 1)$ and $N \in \mathbb{N}$ with, for all $n \geq N$,

$$af(\lceil n/b \rceil) \;\leq\; cf(n).$$

Recall that this condition appears in the third case of the Master theorem. In this exercise you'll learn how regularity can be relatively easily establised for common functions using calculus.

Note that if $f(n) \neq 0$ for all $n > N$, and if the limit $\lim_{n\to\infty} f(\lceil n/b \rceil)/f(n)$ exists, then f is regular iff

$$\lim_{n\to\infty} \frac{f(\lceil n/b \rceil)}{f(n)} \;<\; \frac{1}{a}.$$

1. Use this to prove that $f(n) = n$ is regular iff $a < b$ iff $\log_b(a) < 1$.
   (Hint: $n/b \leq \lceil n/b \rceil \leq n/b + 1$.)
2. Show that $f(n) = n^d$, where $d \in (0, \infty)$, is regular iff $\log_b(a) < d$.
3. Show that $f(n) = q^n$, where $q \in (1, \infty)$, is regular for all a and b.
4. Show that $f(n) = \log(n)$ is regular iff $a < 1$.
5. Show that $f(n) = n \log(n)$ is regular iff $a < b$.

Solution.

1. Since $n/b \leq \lceil n/b \rceil \leq n/b + 1$, we have, for all $n \in \{1, 2, \dots\}$,

   $$1/b \;\leq\; \lceil n/b \rceil /n \;\leq\; 1/b + 1/n.$$

   Since both $1/b$ and $1/b + 1/n$ converge to $1/b$ as $n \to \infty$, and these expressions sandwich $\lceil n/b \rceil /n$, we see that $\lim_{n\to\infty} \lceil n/b \rceil /n$ exists, and equals $1/b$. Whence f is regular iff $1/b < 1/a$, iff $a < b$, iff $\log_b(a) < 1$.

2. Since $f(\lceil n/b \rceil)/f(n) = (\lceil n/b \rceil /n)^d$, and we already know from the previous point that $\lceil n/b \rceil /n \to 1/b$ as $n \to \infty$, and $x \mapsto x^d$ is continuous, we see that $\lim_{n\to\infty} f(\lceil n/b \rceil)/f(n) = 1/b^d$, and so f is regular iff $1/b^d < 1/a$ iff $a \leq b^d$ iff $\log_b(a) < 1$.

3. Since $f(\lceil n/b \rceil)/f(n) = q^{-(n - \lceil n/b \rceil)}$ and $n - \lceil n/b \rceil \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} f(\lceil n/b \rceil)/f(n) = 0$. Thus f is regular iff $0 < 1/a$, that is, always.

4. Since for $n > b$ we have $1 + \log(b)/\log(n) = \log(n/b)/\log(n) \leq f(\lceil n/b \rceil)/f(n) \leq \log(2n/b)/\log(n) = 1 + \log(2b)/\log(n)$, and the outer expressions both converge to 1 as $n \to \infty$, we see that $\lim_{n\to\infty} f(\lceil n/b \rceil)/f(n) = 1$, and so f is regular iff $1 < 1/a$ iff $a < 1$.

5. Since we have already computed the limits $\lim_{n\to\infty} \lceil n/b \rceil / n = 1/b$ and $\lim_{n\to\infty} \log(\lceil n/b \rceil)/\log(n) = 1$, we may take their product, yielding

$$\lim_{n\to\infty} \frac{\lceil n/b \rceil \log(\lceil n/b \rceil)}{n \log(n)} = 1/b \cdot 1 = 1/b.$$

Whence f is regular iff $1/b < 1/a$ iff $a < b$.

■

Grading. **4 points** for each of the five tasks.

(Note the level of rigour in the solution: this exercise is about applying calculus to a complexity problem, not about analysing calculus itself.)

Exercise 3 (50 points) Let numbers $b \in \{2, 3, \dots\}$, and $a \in [1, \infty)$, and functions $T, f \colon \{1, b, b^2, \dots\} \to [0, \infty)$ with

$$T(n) \;=\; aT(n/b) \,+\, f(n) \qquad \text{and} \qquad f(n) \;=\; n^{\log_b(a)} \log(n)$$

for all $n \in \{b, b^2, b^3, \dots\}$ be given. Show that $T(n) = \Theta(n^{\log_b(a)} \log(n)^2)$.

Hint: follow these steps:

1. Draw a recursion tree to deduce (or by induction proof) that, for all $k \in \mathbb{N}$,

$$T(b^k) \;=\, a^k\, T(1) \;+\; \sum_{i=0}^{k-1} a^i f(b^{k-i}).$$

2. Show that, for all $k \in \mathbb{N}$, the following two equalities hold.

$$\sum_{i=0}^{k-1} a^i f(b^{k-i}) \;=\; \log(b)\, a^k \sum_{i=0}^{k-1} (k-i) \;=\; \frac{1}{2} \log(b)\, a^k\, k(k+1)$$

(You already know a formula for $\sum_{i=0}^{k-1}(k-i)$.)

3. Show that $a^k = n^{\log_b(a)}$ for all k, where $n := b^k$.

4. Show that, for all $n \in \{1, b, b^2, \dots\}$,

$$T(n) \;=\; n^{\log_b(a)} \Big(\, T(1) \;+\; \frac{1}{2} \log(b)\, \log_b(n)\, (\log_b(n) + 1) \,\Big).$$

Solution. Writing $n := b^k$ the recursion tree needed here is essentially the same as the one drawn up in §4.6.1 of the book on page 99, but with a^k instead of $n^{\log_b(a)}$ (see third hint), and $T(1)$ instead of $\Theta(1)$. It will be an a-branching tree of height $\log_b(n) = k$ with labels $f(a/b^i)$ at depth $i < k$ and labels $T(1)$ at depth k. Summing all labels yields that, for all $k \in \mathbb{N}$,

$$T(b^k) \;=\; a^k T(1) + \sum_{i=0}^{k-1} a^i f(b^{k-i}). \tag{2}$$

Equation (2) can also be obtained by induction, as follows. For $k = 0$, (2) becomes $T(1) = T(1) + 0$, which is clearly true. Now suppose that (2) holds for some k, then we have

$$\begin{aligned}
T(b^{k+1}) &= aT(b^k) + f(b^{k+1}) && \text{by assumption}\\
&= a^{k+1}T(1) + \Big(\sum_{i=0}^{k-1} a^{i+1} f(b^{k-i})\Big) + f(b^{k+1}) && \text{by the induction hypothesis}\\
&= a^{k+1}T(1) + \Big(\sum_{j=1}^{k} a^{j} f(b^{k+1-j})\Big) + f(b^{k+1}) && \text{by setting } j := i+1\\
&= a^{k+1}T(1) + \sum_{j=0}^{k} a^{j} f(b^{k+1-j}),
\end{aligned}$$

and so we see that (2) holds for $k+1$ too. Conclusion: (2) holds for all $k \in \mathbb{N}$.

We proceed by simplifying $\sum_{i=0}^{k-1} a^i f(b^{k-i})$, as follows. Given $k \in \mathbb{N}$,

$$\begin{aligned}
\sum_{i=0}^{k-1} a^i f(b^{k-i}) &= \sum_{i=0}^{k-1} a^i (b^{k-i})^{\log_b(a)} \log(b^{k-i}) && \text{defn. of } f\\
&= \sum_{i=0}^{k-1} a^i (b^{\log_b(a)})^{k-i} (k-i) \log(b)\\
&= \log(b) \sum_{i=0}^{k-1} a^i a^{k-i} (k-i)\\
&= \log(b)\, a^k \sum_{i=0}^{k-1} (k-i)\\
&= \log(b)\, a^k \sum_{j=0}^{k} j && \text{taking } j := k-i\\
&= \frac{1}{2} \log(b)\, a^k (k+1)k.
\end{aligned} \tag{3}$$

Note that given $k \in \mathbb{N}$, we have $a^k = b^{\log_b(a^k)} = b^{k \log_b(a)} = (b^k)^{\log_b(a)} = n^{\log_b(a)}$ when $n := b^k$, so, when $n \in \{1, b, b^2, \dots\}$ is given, we have

$$\begin{aligned}
T(n) &= T(b^k)\\
&= a^k T(1) + \sum_{i=0}^{k-1} a^i f(b^{k-i}) && \text{by (2)}\\
&= a^k T(1) + \frac{1}{2} \log(b)\, a^k (k+1)k && \text{by (3)}\\
&= n^{\log_b(a)} T(1) + \frac{1}{2} \log(b)\, n^{\log_b(a)} (\log_b(n) + 1) \log_b(n)\\
&= \Theta(n^{\log_b(a)}) + \Theta(n^{\log(a)})\, \Theta(\log(n))\, \Theta(\log(n))\\
&= \Theta(n^{\log_b(a)}) + \Theta(n^{\log_b(a)} \log(n)^2) = \Theta(n^{\log_b(a)} \log(n)^2),
\end{aligned}$$

using here that $n^{\log_b(n)} \in \mathcal{O}(n^{\log_b(a)} \log(n)^2)$. ■

Grading. If the hints are followed, award **10 points** for executing each of the four hints successfully, and **10 points** for the conclusion. If the hints are not followed to the letter, use your best judgement to allot the **40 points**.

Exercise 4 (10 points, extra difficult) Let $b \in \{2, 3, \cdots\}$, $a \in [1, \infty)$, and functions $f, T\colon \mathbb{N} \to [0, \infty)$ with $f(n) = \mathcal{O}(n^{\log_b(a)} \log(n))$, and

$$T(n) \leq aT(\lceil n/b \rceil) + f(n)$$

for all $n \in \{2, 3, \dots\}$ be given. Show that $T(n) = \mathcal{O}(n^{\log_b(a)} \log(n)^2)$.

Solution. By drawing a recursion tree we see that, for all $n \in \{1, 2, \dots\}$,

$$T(n) \leq T(1) \cdot a^{\lceil \log_b(n) \rceil} + \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^k f(\lceil n/b^k \rceil), \tag{4}$$

but it's also easy to proof using induction over $\lceil \log_b(n) \rceil$ using:

Claim 1. $\lceil \log_b(\lceil n/b \rceil) \rceil = \lceil \log_b(n) \rceil - 1$ *for all* $n \in \{1, 2, \dots\}$.

Proof. Let $m \in \mathbb{Z}$ be given. We have $\lceil \log_b(\lceil n/b \rceil) \rceil \leq m$ iff $\log_b(\lceil n/b \rceil) \leq m$ iff $\lceil n/b \rceil \leq b^m$ iff $n/b \leq b^m$ iff $n \leq b^{m+1}$ iff $\log_b(n) \leq m+1$ iff $\lceil \log_b(n) \rceil \leq m+1$ iff $\lceil \log_b(n) \rceil - 1 \leq m$. Thus $\lceil \log_b(\lceil n/b \rceil) \rceil$ and $\lceil \log_b(n) \rceil - 1$ being below themselves must be below each other, making them equal. ■

Back to proving inequality (4). For the base case, note that if $\lceil \log_b(n) \rceil = 0$ for some $n \in \{1, 2, \dots\}$, we must have $n = 1$, and so (4) amounts to $T(1) \leq T(1) + 0$, which is, of course, true.

Suppose now that $\ell \in \{0, 1, 2, \dots\}$ is such that (4) holds for all $n \in \{1, 2, \dots\}$ with $\lceil \log_b(n) \rceil = \ell$; we'll show that (4) holds for all $n \in \{1, 2, \dots\}$ with $\lceil \log_b(n) \rceil = \ell + 1$. So let n with $\lceil \log_b(n) \rceil = \ell + 1$ be given. Then $\lceil \log_b(n) \rceil \geq 1$, so $\log_b(n) > 0$, so $n > 1$, so $T(n) \leq aT(\lceil n/b \rceil) + f(n)$. Moreover, $\lceil \log_b(\lceil n/b \rceil) \rceil = \lceil \log_b(n) \rceil - 1 = \ell$, means that (4) can be applied to $\lceil n/b \rceil$. This all leads to:

$$\begin{aligned}
T(n) &\leq a\,T(\lceil n/b \rceil) + f(n) \\
&\leq T(1) \cdot a^{\lceil \log_b(\lceil n/b \rceil) \rceil + 1} + \sum_{k=0}^{\lceil \log_b(\lceil n/b \rceil) \rceil - 1} a^{k+1} f(\lceil \lceil n/b \rceil / b^k \rceil) + f(n) \\
&= T(1) \cdot a^{\lceil \log_b(n) \rceil} + \sum_{k=0}^{\lceil \log_b(n) \rceil - 2} a^{k+1} f(\lceil n/b^{k+1} \rceil) + f(n) \\
&= T(1) \cdot a^{\lceil \log_b(n) \rceil} + \sum_{k=1}^{\lceil \log_b(n) \rceil - 1} a^{k} f(\lceil n/b^{k} \rceil) + f(n) \\
&= T(1) \cdot a^{\lceil \log_b(n) \rceil} + \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^{k} f(\lceil n/b^{k} \rceil).
\end{aligned}$$

Whence inequality (4) holds for all $n \in \{1, 2, \dots\}$.

Since $a^{\lceil \log_b(n) \rceil} \le a^{\log_b(n)+1} = a \cdot a^{\log_b(n)} = a \cdot n^{\log_b(a)} = \mathcal{O}(n^{\log_b(a)} \log(n)^2)$, we see from (4) that to show $T(n) = \mathcal{O}(n^{\log_b(a)} \log(n)^2)$ it suffices to prove that

$$\sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^k f(\lceil n/b^k \rceil) \;=\; \mathcal{O}(n^{\log_b(a)} \log(n)^2). \tag{5}$$

Since $f(n) = \mathcal{O}(n^{\log_b(a)} \log(n))$ there is $D \in [0, \infty)$ and $N \in \{1, 2, \dots\}$ such that $f(n) \le Dn^{\log_b(a)} \log(n)$ for all $n \ge N$ — which will in fact hold for all $n \in \{2, 3, 4, \dots\}$, by choosing D larger than $\max_{n=1}^{N-1} \frac{f(n)}{n^{\log_b(a)} \log(n)}$.

We'd like to apply this bound for $f(n)$ to (5), but in order to do so, we must first verify that $\lceil n/b^k \rceil \ge 2$ for all $k \le \lceil \log_b(n) \rceil - 1$. Since $\lceil n/b^k \rceil$ will be the smallest for $k = \lceil \log_b(n) \rceil - 1$, it suffices to consider only this particular k. Note that $\lceil n/b^k \rceil \ge 2$ iff $n/b^k > 1$ iff $n > b^k$ iff $\log_b(n) > k$ iff $\log_b(n) + 1 > k + 1 = \lceil \log_b(n) \rceil$, which is indeed so (since $\lceil x \rceil < x + 1$ for any $x \in \mathbb{R}$.)

Since $f(n) \le Dn^{\log_b(a)} \log(n)$ for all $n \ge 2$, we have, writing $\gamma := \log_b(a)$,

$$\begin{aligned} &\sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^k f(\lceil n/b^k \rceil) \\ \le\; & D \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^k \lceil n/b^k \rceil^\gamma \log(\lceil n/b^k \rceil) \end{aligned}$$

Using that $\lceil x \rceil \le 2x$ for all $x \ge \frac{1}{2}$, we get:

$$\begin{aligned} \le\; & D 2^\gamma \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} a^k (n/b^k)^\gamma \log(\lceil n/b^k \rceil) \\ =\; & D 2^\gamma n^\gamma \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} \log(\lceil n/b^k \rceil) && \text{since } b^\gamma = a \end{aligned}$$

Note that $n \le b^{\lceil \log_b(n) \rceil}$, so $n/b^k \le b^{\lceil \log_b(n) \rceil - k}$, so $\log(\lceil n/b^k \rceil) \le \log(b)\,(\lceil \log_b(n) \rceil - k)$, giving:

$$\begin{aligned} \le\; & D 2^\gamma \log(b)\, n^\gamma \sum_{k=0}^{\lceil \log_b(n) \rceil - 1} \lceil \log_b(n) \rceil - k \\ \le\; & D 2^\gamma \log(b)\, n^\gamma \sum_{k'=0}^{\lceil \log_b(n) \rceil} k' && \text{setting } k' := \lceil \log_b(n) \rceil - 1 - k \\ =\; & \mathcal{O}(\, n^\gamma \log(n)^2 \,), \end{aligned}$$

by the well-known formula $\sum_{i=0}^{m} i = \frac{1}{2} m(m+1)$. Hence we have established (5), and thus $T(n) = \mathcal{O}(n^{\log_b(a)} \log(n)^2)$. ■