Complexity exercise set #3

for the tutorial on
April 28, 2022

Exercises marked with an asterisk ($*$) may be handed in for grading and can earn you a small bonus[1] on the exam, provided you submit your solutions via Brightspace in PDF before **15:15 on Monday May 9**.

Exercise 1 Solve problem 3-2 from the book (on page 61 of the third edition.)

Exercise 2 ($*$) **(40 points)** This exercise tests your mathematical exactness: be precise and detailed.

1. Show that $n \leq 2^n$ for all $n \in \{0, 1, 2, \dots\}$ using induction. State the induction hypothesis, and base case(s) explicitly.

   Base Case: Clearly $n \leq 2^n$ holds for $n = 0$, because $0 \leq 2^0 = 1$.

   Inductive Step: Suppose that $n \leq 2^n$ holds for some $n \in \{0, 1, \dots\}$ — this is the *induction hypothesis (IH)*. We must show that $n + 1 \leq 2^{n+1}$. Indeed,

   $$n + 1 \overset{\text{IH}}{\leq} 2^n + 1 \leq 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

   where we have used that $1 \leq 2^m$ for all $m \in \mathbb{N}$. (If you like, this can be proven with induction too: clearly $1 \leq 2^0 \equiv 1$, and given $m \in \mathbb{N}$ with $1 \leq 2^m$, we have $1 \leq 2^m \leq 2 \cdot 2^m = 2^{m+1}$ too.)

   Conclusion: $n \leq 2^n$ holds for all $n \in \mathbb{N}$.

   Grading. **5 points** for at least one base case. (Treating both $n = 0$ and $n = 1$ separately is fine, and has the added bonus of giving an argument for $1 \leq 2^n$, using the induction hypothesis, namely $1 \leq n \leq 2^n$.)

   5 points for the inductive step. It's important that they explicitly state the induction hypothesis somehow (e.g. "assume $n \leq 2^n$ for some $n \in \mathbb{N}$"), but they needn't call it by its name, 'induction hypothesis'.

2. What is meant by the statement "$n = \mathcal{O}(2^n)$"?
   Give a formal definition, and a proof.

[1] For more details, see `https://cs.ru.nl/~awesterb/teaching/2022/complexity.html`.

The notation "$n = \mathcal{O}(2^n)$" means that there is an $N \in \mathbb{N}$ and $C > 0$ such that $n \leq C \cdot 2^n$ for all $n \geq N$. Choosing $N = 0$ and $C = 1$, we get the statement we just proved in point 1, so $n = \mathcal{O}(2^n)$ holds.

More details: The notation $n = \mathcal{O}(2^n)$ is an abbreviation for $n \in \mathcal{O}(2^n)$, which in turn is a shorthand for $f \in \mathcal{O}(g)$ where $f, g \colon \mathbb{N} \to [0, \infty)$ are defined by $f(n) = n$ and $g(n) = 2^n$ for all $n \in \mathbb{N}$. The set $\mathcal{O}(g)$ consists of all functions $h \colon \mathbb{N} \to [0, \infty)$ that are *asymptotically bounded* by g, that is, for which there are $C \in (0, \infty)$ and $N \in \mathbb{N}$ with $h(n) \leq Cg(n)$ for all $n \in \mathbb{N}$ with $n \geq N$. See Section 3.1 of the book.

Grading. **5 points** for defining "$n = \mathcal{O}(2^n)$" correctly, where 'correctly' means that its obvious generalisation to $f = \mathcal{O}(g)$ is equivalent to the definition given above. (For example, $\exists C \geq 0, N > 1 \forall n > N[n \leq C2^n]$ is fine too, but $\forall C \geq 0 \exists N \forall n > N[n \leq C2^n]$ is not.)

5 points for realising we just proved $n = \mathcal{O}(2^n)$ in the previous exercise (or for giving another proof.)

3. Induction may be formulated in terms of sets as follows.

$$\forall A\,[\; 0 \in A \;\wedge\; \forall n \in \mathbb{N}\,[\, n \in A \implies n+1 \in A\,] \implies \mathbb{N} \subseteq A\;]. \tag{1}$$

(Here A ranges over all sets.)

Using (1), prove the following 'strong induction principle'.

$$\forall A[\; \forall n \in \mathbb{N}\,[\forall m \in \mathbb{N}\,[\, m < n \implies m \in A\,] \implies n \in A\,] \implies \mathbb{N} \subseteq A\;].$$

Hint: Apply (1) to $A' := \{\, n \in \mathbb{N} \colon\; \forall m \in \mathbb{N}\,[\, m < n \implies m \in A\,]\,\}$.

Let a set A be given, and define

$$A' \;:=\; \{\, n \in \mathbb{N} \colon\; \forall m < n\,[\, m \in A\,]\,\}.$$

So A' consists of all natural numbers n such that all numbers m smaller than n are in A. So if $A = \{0, 1, 2, 4\}$, then $A' = \{0, 1, 2, 3\}$.

Now, assume that

$$\forall n \in \mathbb{N}\,[\,\forall m \in \mathbb{N}\,[\, m < n \implies m \in A\,] \implies n \in A\,]. \tag{2}$$

Our task is to show that $\mathbb{N} \subseteq A$. Note that (2) means that to show that a natural number n is in A, it suffices to show that all strictly smaller numbers are in A. In terms of the A' we just defined, (2) becomes

$$\forall n \in \mathbb{N}\,[\, n \in A' \implies n \in A\,]. \tag{3}$$

Instead of proving $\mathbb{N} \subseteq A$ directly, we are going to first show that $\mathbb{N} \subseteq A'$, using (1).

(a) $0 \in A'$: this is indeed the case, because there is no $m < 0$, and so $\forall m < 0\,[\, m \in A\,]$ holds 'vacuously'.

(b) $\forall n \in \mathbb{N} [\, n \in A' \implies n+1 \in A' \,]$: Indeed, let $n \in \mathbb{N}$ with $n \in A'$ be given; we must show that $n+1 \in A'$. By (3), we have $n \in A$ (because $n \in A'$). But now we not just have that all $m < n$ are in A (which is what $n \in A'$ means) but also $n \in A$, so that all $m < n+1$ are in A, which means $n+1 \in A'$.

Hence (1) gives us that $\mathbb{N} \subseteq A'$. From this it easily follows that $\mathbb{N} \subseteq A$. Indeed, let $n \in \mathbb{N}$ be given, then $n+1 \in \mathbb{N} \subseteq A'$, so $\forall m < n+1 \,[\, m \in A \,]$, and so in particular (choosing $m = n$) we get $n \in A$. Whence $\mathbb{N} \subseteq A$.

Grading. **5 points** for being clear and **5 points** for being correct. The proof needn't be as detailed as the one above, but for full points it must be apparent that the student understands what they are doing.

4. Given a function $T\colon \mathbb{N} \to [0, \infty)$ such that

$$T(n) \;\leq\; 2T(\lfloor n/2 \rfloor)$$

for all $n \in \{1, 2, \dots\}$, show that $T(n) = \mathcal{O}(n)$.

Hint: Apply the strong induction principle to

$$A := \{\, n \in \mathbb{N}\colon\; T(n+1) \;\leq\; T(1) \cdot (n+1) \,\}.$$

Let A be defined as in the hint. To prove that $T(n) = \mathcal{O}(n)$, it suffices to show that $\mathbb{N} \subseteq A$, because then $\exists N \in \mathbb{N},\, C > 0\; \forall n \geq N \;[\, T(n) \leq Cn \,]$ (taking $N = 1$, and $C = T(1) + 1$.)

We prove $\mathbb{N} \subseteq A$ by strong induction. Let $n \in \mathbb{N}$ with $\{0, \cdots, n-1\} \subseteq A$ be given; we must show that $n \in A$, that is, that $T(n+1) \leq T(1) \cdot (n+1)$.

If $n = 0$, then this becomes $T(1) \leq T(1)$, which is quite true, so we may assume that $n > 0$.

Obviously, we want to apply the induction hypothesis, to $m := \lfloor \frac{n+1}{2} \rfloor - 1$, but before we can do this we must first check that $0 \leq m < n$. To begin, since $n > 0$, we have $n+1 \geq 2$, so $\frac{n+1}{2} \geq 1$, thus $\lfloor \frac{n+1}{2} \rfloor \geq 1$, and so $m \geq 0$. To see that $m < n$, note that $\lfloor \frac{n+1}{2} \rfloor \leq \frac{n+1}{2} \leq n+1$.

Whence $m \in \{0, \dots, n-1\} \subseteq A$, and thus $T(\lfloor \frac{n+1}{2} \rfloor) = T(m+1) \leq T(1) \cdot (m+1) = T(1) \cdot \lfloor \frac{n+1}{2} \rfloor$ — 'by the induction hypothesis'.

$$\begin{aligned}
T(n+1) \;&\leq\; 2T(\lfloor (n+1)/2 \rfloor) && \\
&\leq\; 2T(1)\, \lfloor (n+1)/2 \rfloor && \text{by the IH} \\
&\leq\; T(1)\, \lfloor n+1 \rfloor && \text{since } 2\,\lfloor x \rfloor \leq \lfloor 2x \rfloor \\
&\leq\; T(1)(n+1) && \text{since } n+1 \text{ is whole.}
\end{aligned}$$

Hence $n \in A$. Thus $\mathbb{N} \subseteq A$, by strong induction, and so we're done.

Grading. **5 points** for the essential steps, and **5 points** for getting all the details right.

Exercise 3 $(*)$ **(50 points)** Let $T\colon \mathbb{N} \to [0, \infty)$ be given.

1. Show that $T(n) = \Theta(n)$, when, for all $n \in \{37, 38, 39, \dots\}$

$$T(n) \;=\; 2T(\lfloor n/3 \rfloor + 3) \;+\; 2T(\lceil n/6 \rceil + 4) + 5.$$

Hint: make use of the substitution method from §4.3 of the book.

To show that $T(n) = \Theta(n)$, we must show that $T(n) = \mathcal{O}(n)$ and $T(n) = \Omega(n)$.

We'll start with $T(n) = \mathcal{O}(n)$. Let $d, c \in \mathbb{R}$ be arbitrary. We want to find values for c and d such that $T(n) \leq cn + d$ for all $n \geq 37$, (and if that's not possible, for all $n > N$ for some $N \geq 37$.) Let $n \geq 37$ be given, and suppose (the 'induction hypothesis') that $T(m) \leq cm + d$ for all $m < n$. We want to see what's needed to get $T(n) \leq cn + d$. We have:

$$T(n) \;\leq\; 2T(\lfloor n/3 \rfloor + 3) \;+\; 2T(\lceil n/6 \rceil + 4) \;+\; 5 \qquad \text{since } n \geq 37$$

Note that $\lfloor n/3 \rfloor + 3$ and $\lceil n/6 \rceil + 4$ are both strictly smaller than n, because $n \geq 37$, so we can apply the induction hypothesis to get:

$$\begin{aligned} &\leq\; 2c(\lfloor n/3 \rfloor + 3) \;+\; 2d \;+\; 2c(\lceil n/6 \rceil + 4) \;+\; 2d \;+\; 5 \\ &=\; 2c(\lfloor n/3 \rfloor + \lceil n/6 \rceil + 7) \;+\; 5 \;+ 4d \end{aligned}$$

Since $\lfloor n/3 \rfloor \leq n/3$ and $\lceil n/6 \rceil \leq n/6 + 1$, we get:

$$\begin{aligned} &\leq\; 2c(n/3 + n/6 + 8) \;+\; 5 \;+ 4d \\ &=\; 2c(n/2 + 8) \;+\; 5 \;+ 4d \\ &=\; cn \;+\; 16c + 5 + 4d \end{aligned}$$

so *if* $16c + 5 + 4d \leq d$, *then* we get:

$$\leq\; cn \;+\; d,$$

and we're good, well, at least for the induction step.

Hence we define $d := -(16c + 5)/3$, so that we have $16c + 5 + 4d \leq d$.

There is, however, a catch: while we assumed that $T(m) \leq cm + d$ for all $m < n$, this won't be true for small m, because $cm + d = c(m - 16/3) - 5/3 < 0$ for $m < 16/3$, irrespective of the value of c. One might worry that this would prevent us from establishing a base case for our induction.

Upon closer inspection, however, the failure for small m does not matter. Indeed, note that $m - 16/3 \geq 0$ for all $m \geq 6$. We *can* choose c so large that $T(m) \leq c(m - 16/3) - 5/3$ for all $m \in \{6, 7, \dots, 36\}$ (by letting c be the maximum of $(3T(m) + 5)/(3m - 16)$ for $m \in \{6, 7, \dots, 36\}$.) We thus have not one, but multiple base cases for our induction.

Now that everything is in place, let's show that $T(n) = \mathcal{O}(n)$. We will prove that

$$T(n) \;\leq\; c(n - 16/3) - 5/3 \tag{4}$$

for all $n \geq 6$, using strong induction. So let $n \geq 6$ be given such that (4) holds for all m with $6 \leq m < n$; we must show that (4) holds for n

as well. If $n < 36$, then we know that (4) holds by choice of c, and so we're done. So let's assume that $n \geq 37$. As we saw before, we get $T(n) \leq cn + d \equiv c(n - 16/3) - 5/3$ provided that $T(m) \leq cm + d$ holds for $m = \lfloor n/3 \rfloor + 3$ and $m = \lceil n/6 \rceil + 4$. By the induction hypothesis, this is indeed the case *provided* that $m \geq 6$. This will indeed be the case, since

(a) $\lfloor n/3 \rfloor + 3 < 6$ implies $\lfloor n/3 \rfloor < 3$ implies $n/3 \leq \lfloor n/3 \rfloor + 1 < 4$ implies $n \leq 12$, while $n \geq 37$, and

(b) $\lceil n/6 \rceil + 4 < 6$ implies $\lceil n/6 \rceil < 2$ implies $n/6 \leq 2$ implies $n \leq 12$, while $n \geq 37$.

Hence we get $T(n) \leq cn + d$, for all $n \geq 6$. Thus $T(n) \in \mathcal{O}(n)$.

It remains to be shown that $T(n) = \Omega(n)$. Again, let $c, d \in \mathbb{R}$ for now be arbitrary, and let $n \geq 37$ with $T(m) \geq cm + d$ for all $m < n$ be given. We want to see what's needed to get $T(n) \geq cn + d$. Note that

$$T(n) \; = \; 2T(\lfloor n/3 \rfloor + 3) \; + \; 2T(\lceil n/6 \rceil + 4) \; + \; 5$$

Using the induction hypothesis:

$$\geq \; 2c(\lfloor n/3 \rfloor + 3 + \lceil n/6 \rceil + 4) \; + \; 4d \; + \; 5$$

Using the fact that $\lfloor n/3 \rfloor \geq n/3 - 1$ and $\lceil n/6 \rceil \geq n/6$:

$$\begin{aligned} &\geq \; 2c(n/3 + 2 + n/6 + 4) \; + \; 4d \; + \; 5 \\ &\geq \; cn \; + \; 12c \; + \; 4d \; + \; 5 \end{aligned}$$

so defining $d := -4c - 5/3$, we get:

$$= \; cn \; + \; d.$$

This covers the n with $n \geq 37$. For the induction to 'get off the ground' it suffices to find $c \in (0, \infty)$ such that

$$T(n) \geq cn - 4c - 5/3 \qquad \text{for all } n \in \{0, \ldots, 36\},$$

for then the reasoning above gives us $T(n) \geq cn - 4c - 5/3$ for all $n \in \mathbb{N}$. Define $c := \min_{n<37}(T(n) + 5/3)/(n+1)$. Then $c > 0$, and $cn - 4c - 5/3 \leq c(n+1) - 5/3 \leq T(n) + 5/3 - 5/3 = T(n)$ for all $n < 37$.

Whence $T(n) \geq cn - 4c - 5/3$ for all $n \in \mathbb{N}$. In particular, $T = \Omega(cn - 4c - 5/3) = \Omega(n)$.

Grading. **15 points** for $T(n) = \mathcal{O}(n)$ and **10 points** for $T(n) = \Omega(n)$. Deduct at most *5 points* in total for not dealing with the base cases correctly.

2. Show that $T(n) = \mathcal{O}(\log(\log(n)))$ when, for all $n \in \{2, 3, \ldots\}$,

$$T(n) \; = \; T(\lfloor \sqrt{n} \rfloor) + 5.$$

Note that $\log(\log(n)) > 0$ for all $n \geq 4$, so we can find $C > 5/\log(2)$ such that $T(n) \leq C\log(\log(n))$ for all $n \in \{4, 5, \ldots, 15\}$. We'll show by strong induction that, for all $n \geq 4$,

$$T(n) \;\leq\; C\log(\log(n)). \tag{5}$$

So let $n \geq 4$ be given such that $T(m) \leq C\log(\log(m))$ for all $m \in \mathbb{N}$ with $4 \leq m < n$. If $n < 16$, then we know that (5) holds, by choice of C, so assume that $n \geq 16$. Note that then $\lfloor\sqrt{n}\rfloor \geq 4$, so:

$$\begin{aligned}
T(n) &= T(\lfloor\sqrt{n}\rfloor) \;+\; 5 \\
&\leq C\log(\log(\lfloor\sqrt{n}\rfloor)) \;+\; 5 && \text{by the IH} \\
&\leq C\log(\log(\sqrt{n})) \;+\; 5 && \text{since } \lfloor\sqrt{n}\rfloor \leq \sqrt{n} \\
&= C\log(1/2\log(n)) \;+\; 5 && \text{since } \sqrt{n} = n^{1/2} \\
&= C\log(1/2) \;+\; C\log(\log(n)) \;+\; 5 \\
&= C\log(\log(n)) \;+\; (5 - C\log(2)) \\
&\leq C\log(\log(n)),
\end{aligned}$$

because $5 \leq C\log(2)$ since $C \geq 5/\log(2)$. Hence $T(n) = \mathcal{O}\left(\log(\log(n))\right)$.

Grading. **5 points** for properly taking care of the base case(s) and **20 points** for the inductive step.

Exercise 4 $(*)$ **(10 points, difficult)** Match each recurrence relation to the correct asymptotic solution.

1. $T(n) = 4T(\lceil n/3\rceil) + n\lg(n)$	i $\Theta(n\log(\log(n)))$
2. $T(n) = 3T(\lfloor n/3\rfloor) + n/\lg(n)$	ii $\Theta(n)$
3. $T(n) = 4T(\lceil n/2\rceil) + n^2\sqrt{n}$	iii $\Theta(n\log(n))$
4. $T(n) = 3T(\lceil n/3\rceil - 2) + n/2$	iv $\Theta(\mathrm{li}(n))$
5. $T(n) = 2T(\lceil n/2\rceil) + n/\lg(n)$	v $\Theta(n\log(\log(n)))$
6. $T(n) = T(\lfloor n/2\rfloor) + T(\lfloor n/4\rfloor) + T(\lfloor n/8\rfloor) + n$	vi $\Theta(n^{\log_3(4)})$
7. $T(n) = T(n-1) + 1/n$	vii $\Theta(\log(n))$
8. $T(n) = T(n-1) + \lg(n)$	viii $\Theta(n^{2\frac{1}{2}})$
9. $T(n) = T(n-2) + 1/\lg(n)$	ix $\Theta(n\log(\log(n)))$
10. $T(n) = \sqrt{n}\,T(\lfloor\sqrt{n}\rfloor) + n$	x $\Theta(n\log(n))$

The correct relation between recurrence and solution is as follows.

1	vi
2, 5, 10	i, v, ix
3	viii
4, 8	iii, x
6	ii
7	vii
9	iv

1. $T(n) = 4T(\lceil n/3 \rceil) + n\lg(n)$ has solution $\Theta(n^{\log_3(4)})$, by the Master Theorem, case I, since $\log_3(4) > 1$.

2. $T(n) = 3T(\lfloor n/3 \rfloor) + n/\lg(n)$ has solution $\Theta(n\log(\log(n)))$.

   By exercise 4 of exercise set #2, we know that $T(n) = \mathcal{O}(n\log(\log(n)))$. On the other hand, we have $T(n) = \Omega(n)$, because T will be bounded below by some solution of $T'(n) = 3T(\lfloor n/3 \rfloor) + \sqrt{n}/\lg(n)$ (which by the Master Theorem is in $\Theta(n)$) since $\sqrt{n}/\lg(n) \leq n/\lg(n)$.

   This leaves only $\Theta(n\log(\log(n)))$ (i, v, ix), and $\Theta(n)$ (ii) as options, and since we'll see that ii is taken by 6, we conclude that the solution of the present recurrence relation must be $\Theta(n\log(\log(n)))$.

3. $T(n) = 4T(\lceil n/2 \rceil) + n^2\sqrt{n}$ has solution $\Theta(n^{2.5})$, by the Master Theorem, case III, since $\log_2(4) = 2 < 2.5$.

4. $T(n) = 3T(\lceil n/3 \rceil - 2) + n/2$ has solution $\Theta(n\log(n))$.

   This can be seen by defining $S(n) := T(n-3)$, which obeys the recurrence $S(n) = 3S(\lceil n/3 \rceil) + (n-3)/2$, and thus has solution $\Theta(n)$, by case 2 of the Master Theorem. Whence $T(n) = S(n+3) = \Theta(n+3) = \Theta(n)$.

5. $T(n) = 2T(\lceil n/2 \rceil) + n/\lg(n)$ has solution $\Theta(n\log(\log(n))$, by a reasoning similar to the one used for recurrence number 2.

6. $T(n) = T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/8 \rfloor) + n$ has solution $\Theta(n)$.

   Clearly, $T(n) = \Omega(n)$, because $T(n) \geq n$. On the other hand, one can see that $T(n) = \mathcal{O}(n)$, by proving $T(n) \leq Cn$ using strong induction using the inequalities $T(n) = T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor) + T(\lfloor n/8 \rfloor) + n \leq C(\lfloor n/2 \rfloor + \lfloor n/4 \rfloor) + \lfloor n/8 \rfloor) + n \leq C(n/2 + n/4 + n/8 + n/C) = nC(7/8 + 1/C) \leq Cn$ when $C \geq 8$.

7. $T(n) = T(n-1) + 1/n$ has solution $\Theta(\log(n))$.

   First note that $T(n)$ will be equal (up to a constant) to $\sum_{n=1}^{N} 1/n$, so our task is to find the order of this sum.

   In general, when $f\colon \mathbb{R} \to \mathbb{R}$ is a monotonic function (increasing or decreasing), the sum $\sum_{n=M}^{N} f(n)$ will lie between the integrals $\int_{[M-1,N]} f(x)dx$ and $\int_{[M,N+1]} f(x)dx$. In particular, since $\int 1/xdx = \log(n)$, we have $\log(N) - \log(1) \leq \sum_{n=2}^{N} 1/n \leq \log(N+1) - \log(2)$, and so the sum — and thus $T(N)$ too — is in $\Theta(\log(N))$

8. $T(n) = T(n-1) + \lg(n)$ has solution $\Theta(n\log(n))$ by a reasoning similar to the one for recurrence number 8, but now using that sum $\sum_{n=1}^{N} \lg(n)$ is in $\Theta(n\log(n))$, because $\int \log(x)dx = x\log(x) - x$.

9. $T(n) = T(n-2) + 1/\lg(n)$ has solution $\Theta(\mathrm{li}(n))$, where $\mathrm{li}(x) = \int_0^x \frac{dt}{\log(t)}$ is the 'logarithmic integral', by the same reasoning as for recurrences number 8 and 9.

   As formally li has not been defined, one can also keep this recurrence until the end and conclude that its solution must be the only remaining asymptotic solution, being $\Theta(\mathrm{li}(n))$.

10. $T(n) = \sqrt{n}\,T(\lfloor\sqrt{n}\rfloor) + n$ has solution $\Theta(n\log(\log(n))$.

    Consider $S(n) = T(n)/n$. It obeys the recurrence $S(n) \leq S(\lfloor\sqrt{n}\rfloor)+1$, and so by the methods of part 2 of Exercise 3, we get $S(n) = \mathcal{O}(\log(\log(n)))$, and thus $T(n) = \mathcal{O}(n\log(\log(n)))$. Since clearly, $S(n) = \Omega(1)$, we get $T(n) = \Omega(n)$, and so the only options for $\Theta(T(n))$ are $\Theta(n\log(\log(n)))$ (i, v, ix) and $\Theta(n)$ (ii), but sice $\Theta(n)$ has already been taken, that leaves us with $\Theta(n\log(\log(n)))$.