Complexity exercise set #5

for the tutorial on
May 19, 2022

Exercises marked with an asterisk ($*$) may be handed in for grading and can earn you a small bonus[1] on the exam, provided you submit your solutions via Brightspace in PDF before **15:15 on Monday May 30**.

Exercise 1 In this exercise, you get familiar with some notions introduced during the lecture.

1. Give a graph that does not have a clique of size 3.
2. Give a graph that does not have a 3-coloring.
3. Give a graph that has a 3-coloring and a clique of size 3.

Exercise 2$*$ (50 points) In this exercise, we look at *independent sets* of graphs. Suppose, that we have a graph G. An **independent set** of G is a set X of vertices in G such that there is no edge from x to y if both x and y are members of X.

1. Give an example of a graph and two nonempty sets of vertices of which one is an independent set and the other is not.
2. Show that the following decision problem is **NP**-complete:
   Given a graph G and an integer k, is there an independent set with at least k vertices?

Exercise 3$*$ (50 points) In this exercise, we look at the set cover problem. Suppose that we have a finite set $U \subset \mathbb{N}$ and a set S of subsets to U.

1. Take $U = \{1, \ldots, 10\}$ and take

   $$S = \{\{2,4,7\}, \{2,4,8\}, \{1,5,7\}, \{8,9\}, \{1,3,9\}, \{6,10\}, \{1,5,8\}\}$$

   Find a subset S' of S with size 4 such that the union of S' is U.
2. Show that the following problem is **NP**-complete:
   Given an integer k and sets U and S as above, is there a subset S' of S with k elements such that the union of the elements in S' is equal to U?

[1] For more details, see `https://cs.ru.nl/~awesterb/teaching/2022/complexity.html`.

Exercise 4 (more difficult) Let G be a connected graph whose set of vertices is V and whose set of edges is E. A set $X \subseteq V$ is called **dominating** if for every vertex $v \in V$ we either have $v \in X$ or there is an $x \in X$ and an edge between v and x.

1. Give an example of a dominating set that is not a vertex cover
2. Show that the following problem is **NP**-complete:
   Given a graph G and an integer k, is there a dominating set of size at most k?