Complexity exercise set #6

for the tutorial on
June 2, 2022

Exercises marked with an asterisk ($*$) may be handed in for grading and can earn you a small bonus[1] on the exam, provided you submit your solutions via Brightspace in PDF before **15:15 on Monday June 6**.

Exercise 1$*$ (30 points) Let G be a graph and let k be an integer. A path is called **simple** if it touches every vertex at most once, and the **length** of a path is equal to the number of edges in that path. Show that the following decision problem is **NP**-complete:

Is there a simple path in G with length k?

Solution. First, we show that the problem is in **NP**. A certificate is a valid path of the graph. To check that the path is simple we can count the occurrences of each node in the path and return 1 iff no vertex occurs more than once. A simple path y has $|y| = O(|G|)$ by definition, and counting the number of visits to each node is linear in $|y|$. Thus, the problem is in **NP**.

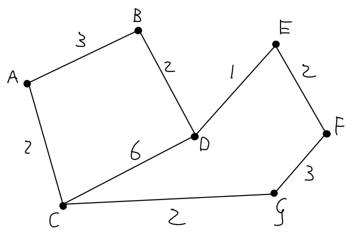
To show that the problem is **NP**-hard, we reduce Ham to it.

Let n be the number of nodes in G. Note that G has a Hamiltonian path if and only if G has a simple path of length n. Certainly, if G has a Hamiltonian path, this path is also simple and of length $n-1$. Conversely, if we have a simple path of length $n-1$, it must necessarily visit every node, as we have taken n to be the number of nodes in G. Hence, we can reduce Ham to this problem by simply taking the reduction which returns the original graph G, together with the number of nodes of G which we can count in time linear in $|G|$ (in fact $|V|$). This is thus polynomial in $|G|$, so we have the required reduction and the given problem is **NP**-complete. ∎

Exercise 2$*$ (35 points) Let G be a graph and let v be a function that assigns an integer to every edge in G. A path is called **simple** if it crosses every vertex at most once.

1. Find a simple path in the following graph such that the sum of all the values in that path is at least 17.

[1] For more details, see `https://cs.ru.nl/~awesterb/teaching/2022/complexity.html`.



2. Given an integer V, show that the following decision problem is **NP**-complete:
   Is there a simple path in G such that the sum of the values of every edge in that path is at least V?

Solution. For the first: A-B-D-C-G-F-E has weight 18

Now we show that the problem is in **NP**. A certificate is a path in G. We have already seen that such a certificate is of the required polynomial size and that checking if this path is simple can be done in polynomial time. Further, summing the values of the edges can be done in time linear in the number of edges in the path and comparing this to the bound V is a constant-time operation. Thus, the problem is in **NP**.

To show that the problem is also **NP**-hard, we can reduce Ham to it. For a graph G, take v to be the function that assigns the value 1 to every edge. Let n be the number of vertices in G. Computing v and n can both be done in time linear in $|E|$ and $|V|$ respectively. Note that there is a Hamiltonian path in G if and only if there is a path in G whose sum of v-values is $n-1$, by the same argument as for the previous exercise where the sum of v-values is the length of the path -1. This gives the desired polynomial reduction, as we take our new graph to be $v(G)$ and let $V = n$, so the problem is **NP**-complete. ■

Exercise 3$*$ (35 points) Let $S \subset \mathbb{N}$ be finite. A **partition** of S is a subset $A \subseteq S$ such that $\sum_{x \in A} x = \sum_{x \in \overline{A}} x$.

1. Find a partition of $\{3, 5, 8, 9, 11, 12\}$
2. Show that the following problem is **NP**-complete:
   Given a set S, is there a partition of S?

Solution. For the first part take, e.g., $A = \{3, 9, 12\}$. Then $\sum_{x \in A} x = 24 = 5 + 8 + 11 = \sum_{x \in \overline{A}} x$.

Next, we show that the problem is in **NP**. A certificate is some subset $A \subseteq S$. To verify this, we sum the elements of A, compute $\overline{A}$, sum the elements of $\overline{A}$ and compare the values for equality. Each step can be done in time polynomial in $|(S, A)|$, so the problem is in **NP**.

To show that the problem is **NP**-hard, we show that SubsSum reduces to it. Given a subset $\{w_1, \ldots, w_n\}$ and an integer t. We can assume that $n > 3$ and that $t > 0$, because if $n = 0$, $n = 1$, $n = 2$ or $t = 0$, we can find the answer in polynomial time.

Now we look at three cases:

1. $2 \cdot t = \sum_{i \in \{w_1,\ldots,w_n\}} i$
2. $2 \cdot t \neq \sum_{i \in \{w_1,\ldots,w_n\}} i$ and $2 \cdot t \neq w_i$ for all i
3. $2 \cdot t = w_i$ for some i

Case 1: Suppose that $2 \cdot t = \sum_{i \in \{w_1,\ldots,w_n\}} i$. In this case, a subset of $\{w_1, \ldots, w_n\}$ with sum t is the same as a partition of this set.

Case 2: If $2 \cdot t \neq \sum_{i \in \{w_1,\ldots,w_n\}} i$ and $2 \cdot t \neq w_i$ for all i, then we define $w' = \sum_{i=1}^{n} w_i$ and we take

$$S = \{w_1, \ldots, w_n, 2 \cdot t, w'\}$$

Since $n > 2$, we have that $w' > w_i$ for every i. Note:

$$\sum_{x \in S} x = 2 \cdot t + w' + \sum_{i=1}^{n} w_i = 2 \cdot (t + w')$$

Hence, a partition S is a subset A with weight $t+w'$. Note that $w' \in A$ or $w' \in \overline{A}$, and without loss of generality, we assume that $w' \in A$. The set $A - \{w'\}$ is then the desired set.

For the converse, note that if we have a set A such that $\sum_{x \in A} x = t$, we can take

$$A' = A \cup \{w'\}$$

and this A' gives the desired partition.

Case 3: Suppose, $2 \cdot t = w_i$ for some i. Without loss of generality, we assume $i = 1$. Note that a subset of $\{w_2, \ldots, w_n\}$ with sum t is also a subset of $\{w_1, \ldots, w_n\}$ with sum t. In addition, every subset of $\{w_1, \ldots, w_n\}$ with sum t is also a subset of $\{w_2, \ldots, w_n\}$. That is because w_1 cannot be included in any subset with sum since $w_1 = 2 \cdot t$ and $t > 0$. Consider the set $\{w_2, \ldots, w_n\}$, and define $w' = \sum_{i \in \{w_2,\ldots,w_n\}} i$. If $w' = 2 \cdot t$, then a subset of $\{w_2, \ldots, w_n\}$ with sum t is the same as a partition. Otherwise, we have $w' \neq 2 \cdot t$, and now the proof goes the same as in the previous case. Note that this works, because we have $n > 3$.

The reduction requires us to compute S, which involves computing $2 \cdot t$ and the sum of n integers. This is polynomial in the size of the original set. Thus, the problem is **NP**-complete. ∎