# Termination and Complexity in Higher-Order Term Rewriting

Part 2. Termination: non-termination

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Download handout and slides from:

https://www.cs.ru.nl/~cynthiakop/2024\_isr/

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a term s is terminating if every reduction sequence starting in s is finite; i.e., there is no infinite reduction sequence
 s →<sub>R</sub> t<sub>1</sub> →<sub>R</sub> t<sub>2</sub> →<sub>R</sub> ...

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### Example:

$$a \rightarrow a$$
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Specialised methods: note the shape of an infinite reduction

$$f(s(0),F) \rightarrow f(0,\lambda y.s(F \cdot y))$$
  
$$f(0,F) \rightarrow f(F \cdot s(0),F)$$

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The primary higher-order difficulty is extending semi-unification techniques.

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 $L : [term \Rightarrow term] \Rightarrow term$ 

 $@ : [term \times term] \Rightarrow term$ 

 $\mathbb{Q}(L(F),x) \rightarrow F \cdot x$ 

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Self-loop: Let  $\omega := L(\lambda x. Q(x, x))$ .

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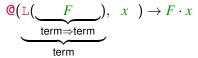
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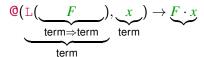
$$\mathbb{Q}(\omega,\omega) \to_{\mathcal{R}} (\lambda x.\mathbb{Q}(x,x)) \cdot \omega \to_{\beta} \mathbb{Q}(\omega,\omega)$$

$$\mathbf{O}(\mathsf{L}( F), x) \to F \cdot x$$

$$\mathbf{O}(\mathbb{L}(\underbrace{F}_{\mathsf{term} \to \mathsf{term}}), x) \to F \cdot x$$







### Finding $\omega\omega$ elsewhere

#### A different example:

```
f :: (A \Rightarrow B \Rightarrow C) \Rightarrow A
g :: A \Rightarrow B \Rightarrow A \Rightarrow C
```

 $h \ :: \ C \Rightarrow C$ 

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f :: (A \Rightarrow B \Rightarrow C) \Rightarrow A
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$$\omega = \lambda x . g(x, , x)$$

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$$g(\omega, z, \omega) \rightarrow_{\mathcal{R}}^{*} h(g(\omega, z, \omega))$$

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```
L :: (\text{term} \Rightarrow \text{term}) \Rightarrow \text{term}

0 :: \text{term} \Rightarrow \text{term} \Rightarrow \text{term}

c :: \text{term} \Rightarrow \text{term}

\text{(L(F),x)} \rightarrow F \cdot \text{c(x)}
```

L :: 
$$(a \Rightarrow b) \Rightarrow b$$
  
O ::  $b \Rightarrow a \Rightarrow b$   
O(L(F),x)  $\rightarrow F \cdot x$ 

• Reduction:  $C[D[F], x] \rightarrow^* E[F \cdot s_1 \cdots x \cdots s_k]$ 

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- Variables:  $F: \sigma_1 \Rightarrow \ldots \Rightarrow \sigma_i \Rightarrow \ldots \Rightarrow \sigma_k \Rightarrow \tau \text{ and } x: \sigma_i$

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- Variables:  $F: \sigma_1 \Rightarrow \ldots \Rightarrow \sigma_i \Rightarrow \ldots \Rightarrow \sigma_k \Rightarrow \tau$  and  $x: \sigma_i$
- $C[D[F], x] : \tau$  and  $D[F] : \sigma_i$

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- Then let  $\omega := D[\lambda x_1 \dots x_k . C[x_i, x_i]]$

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- $C[D[F], x] : \tau$  and  $D[F] : \sigma_i$
- F and x do not appear at other positions in C or D
- Then let  $\omega := D[\lambda x_1 \dots x_k . C[x_i, x_i]]$
- We have:  $C[\omega,\omega] \to^* E[(\lambda x_1 \dots x_k.C[x_i,x_i]) \cdot \omega] \to^*_{\beta} E[C[\omega,\omega]]$

#### **Exercises**

Construct a (general) self-loop for the following HTRSs:

$$f :: 0 \Rightarrow 0 \Rightarrow 0$$

$$g :: 0 \Rightarrow 0$$

$$h :: (0 \Rightarrow 0) \Rightarrow 0$$

$$f(y, h(F)) \rightarrow F \cdot g(y)$$

$$g(x) \rightarrow x$$

$$f :: c \Rightarrow a$$

$$g :: a \Rightarrow c$$

$$h :: (a \Rightarrow b) \Rightarrow c$$

$$k :: a \Rightarrow c \Rightarrow b$$

$$k(f(h(F)), g(y)) \rightarrow F \cdot y$$

#### Bonus exercises

Construct a (general) self-loop for the following HTRSs:

$$f :: a \Rightarrow (a \Rightarrow a)$$

$$g :: (a \Rightarrow a) \Rightarrow a$$

$$\underline{f(g(x))} \rightarrow x$$

$$f :: (b \Rightarrow a \Rightarrow b \Rightarrow a) \Rightarrow c$$

$$g :: b \Rightarrow c$$

$$h :: c \Rightarrow b$$

$$k :: c \Rightarrow b \Rightarrow b \Rightarrow a \Rightarrow a$$

$$k(g(x), y, h(f(F)), z) \rightarrow F \cdot h(g(y)) \cdot z \cdot x$$

```
\max(F,[]) \rightarrow []
\max(F, \cos(x,y)) \rightarrow \cos(F \cdot x, \max(F,y))
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Proof: choose  $\omega := cons(\lambda x_0.map(\lambda y_{0\Rightarrow 0}.\lambda z_0.y_{0\Rightarrow 0}\cdot x_0,x_0))$ . Then:

#### Not terminating if:

[] :: 0  
cons :: 
$$(0 \Rightarrow 0) \Rightarrow 0 \Rightarrow 0$$
  
map ::  $((0 \Rightarrow 0) \Rightarrow 0 \Rightarrow 0) \Rightarrow 0 \Rightarrow 0$ 

Proof: choose  $\omega := cons(\lambda x_0.map(\lambda y_{0\Rightarrow 0}.\lambda z_0.y_{0\Rightarrow 0}\cdot x_0,x_0))$ . Then:

$$\begin{array}{ll} & \max(\lambda y_{0\Rightarrow 0}.\lambda z_{0}.y_{0\Rightarrow 0}\cdot\omega,\omega) \\ \rightarrow & \cos((\lambda y.\lambda z.y\cdot\omega)\langle\lambda x.\mathrm{map}(\lambda y.\lambda z.y\cdot x,x)\rangle,\,\mathrm{map}(\dots)) \\ = & \cos(\lambda z.\langle\lambda x.\mathrm{map}(\lambda y.\lambda z'.y\cdot x,x)\rangle\cdot\omega,\mathrm{map}(\dots)) \\ \rightarrow_{\beta} & \cos(\lambda z.\underline{\mathrm{map}}(\lambda y.\lambda z'.y\cdot\omega,\omega),\,\mathrm{map}(\dots)) \end{array}$$

#### Not terminating if:

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map ::  $((0 \Rightarrow 0) \Rightarrow 0 \Rightarrow 0) \Rightarrow 0 \Rightarrow 0$ 

Proof: choose  $\omega := cons(\lambda x_0.map(\lambda y_{0\Rightarrow 0}.\lambda z_0.y_{0\Rightarrow 0}\cdot x_0,x_0))$ . Then:

(But *is* terminating if cons ::  $(a \Rightarrow a) \Rightarrow o \Rightarrow o$ .)