Termination and Complexity in Higher-Order Term Rewriting

Part 3. Termination: the higher-order recursive path ordering

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ISR 2024

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To start: we will define a well-founded ordering

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$$\begin{array}{rcl} \mathtt{add}(\mathtt{0}, y) & \rightarrow & y \\ \mathtt{add}(\mathtt{s}(x), y) & \rightarrow & \mathtt{s}(\mathtt{add}(x, y)) \end{array}$$

Needed: $\mathtt{add}(\mathtt{0}, \mathtt{0}) \succ \mathtt{0}$, $\mathtt{add}(\mathtt{0}, \mathtt{add}(x, y)) \succ \mathtt{add}(x, y), \ldots$

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- if $s \succ t$ then $s\sigma \succ t\sigma$ for all substitutions σ (we say: $\succ$ is **stable**)

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Such an ordering is called a **reduction ordering**.

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- (lex) $t = \mathtt{f}(t_1, \ldots, t_n)$ and $\mathtt{f}(s_1, \ldots, s_n) \succ_{\mathrm{LPO}} t_i$ for all $i \in \{1, \ldots, n\}$, and $[s_1, \ldots, s_n](\succ_{\mathrm{LPO}})_{\mathrm{lex}}[t_1, \ldots, t_n]$;

LPO example

$$
\begin{array}{rcl}
\mathtt{add}(\mathtt{0}, y) & \rightarrow & y \\
\mathtt{add}(\mathtt{s}(x), y) & \rightarrow & \mathtt{s}(\mathtt{add}(x, y)) \\
\mathtt{mul}(\mathtt{0}, y) & \rightarrow & \mathtt{0} \\
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(lex) $t = \mathtt{f}(t_1, \ldots, t_n)$; $s \succ_{\mathrm{LPO}}$ each t_i; $\vec{s}(\succ_{\mathrm{LPO}})_{\mathrm{lex}}\vec{t}$;

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Exercise

Use LPO to prove termination of the well-known Ackermann function, defined by:

$$
\begin{array}{rcl}
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\mathtt{A}(\mathtt{s}(x), \mathtt{0}) & \rightarrow & \mathtt{A}(x, \mathtt{s}(\mathtt{0})) \\
\mathtt{A}(\mathtt{s}(x), \mathtt{s}(y)) & \rightarrow & \mathtt{A}(x, \mathtt{A}(\mathtt{s}(x), y))
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$$

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Theorem

If $\ell \succ_{\mathrm{LPO}} r$ for all rules in $\mathcal{R}$, then the TRS with rules $\mathcal{R}$ is terminating.

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Proof. $\succ_{\text{LPO}}$ is:

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- monotonic: if $s \succ_{\text{LPO}} t$ then $\mathtt{f}(\ldots, s, \ldots) \succ_{\text{LPO}} \mathtt{f}(\ldots, t, \ldots)$

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- stable: if $s \succ_{\mathrm{LPO}} t$ then $s\sigma \succ_{\mathrm{LPO}} t\sigma$
- monotonic: if $s \succ_{\mathrm{LPO}} t$ then $\mathtt{f}(\dots, s, \dots) \succ_{\mathrm{LPO}} \mathtt{f}(\dots, t, \dots)$
- well-founded: there is no infinite decreasing sequence

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Define: s is terminating if there is no infinite sequence $s \succ_{\mathrm{LPO}} t_1 \succ_{\mathrm{LPO}} t_2 \succ_{\mathrm{LPO}} \ldots$ starting in s.

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Proof: by induction on:

- $\mathtt{f}$ first (using $\rhd$)
- $(s_1, \dots, s_n)$ ordered lexicographically by $\succ_{\text{LPO}}$ second;
- the derivation of $\mathtt{f}(s_1, \dots, s_n) \succ_{\text{LPO}} t$ third

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Conclude: if there is a smallest non-terminating $\mathtt{f}(s_1, \dots, s_n)$, then it must be terminating after all!

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Challenge: mutual recursion

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Solution: allow an **equivalence relation** $\approx$ compatible with $\rhd$, and set $\mathtt{f} \approx \mathtt{g}$

Extending LPO

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Solution: allow some function symbols to order arguments using the **multiset ordering**

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Solution: allow some function symbols to order arguments using the **multiset ordering**

This yields the **recursive path ordering** (RPO).

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Recall: if

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Conclusion:

- A dedicated higher-order definition is needed.
- Types are important!

HOLPO

- $\mathtt{f}(s_1, \ldots, s_n) \sqsupset_{\mathrm{LPO}} t$ if:

  (sub) $s_i \succeq_{\mathrm{LPO}} t$ for some $i \in \{1, \ldots, n\}$

  (copy) $t = \mathtt{g}(t_1, \ldots, t_m)$ and $\mathtt{f} \rhd \mathtt{g}$ and $\mathtt{f}(s_1, \ldots, s_n) \sqsupset_{\mathrm{LPO}} t_j$ for all $j \in \{1, \ldots, m\}$

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- $s \succ_{\text{LPO}} t$ if s and t **have the same type** and:

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Collapsing types in HOLPO

$$\begin{array}{rcl} [] & : & \text{natlist} \\ \mathtt{cons} & : & \text{nat} \Rightarrow \text{natlist} \Rightarrow \text{natlist} \\ \mathtt{map} & : & (\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{natlist} \Rightarrow \text{natlist} \end{array}$$

$$\begin{array}{lcl} \mathtt{map}(F, []) & \rightarrow & [] \\ \mathtt{map}(F, \mathtt{cons}(x, y)) & \rightarrow & \mathtt{cons}(F \cdot x, \mathtt{map}(F, y)) \end{array}$$

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Sometimes problematic: Not $\texttt{cons}(x, y) \succ y$ due to types!

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Solution:

$$\begin{array}{rcl} \mathtt{[]} & : & \text{o} \\ \mathtt{cons} & : & \text{o} \Rightarrow \text{o} \Rightarrow \text{o} \\ \mathtt{map} & : & (\text{o} \Rightarrow \text{o}) \Rightarrow \text{o} \Rightarrow \text{o} \end{array}$$

Example

$$
\begin{array}{rcl}
[] & : & o \\
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\end{array}
$$

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Choose $\mathtt{map} \rhd \mathtt{cons}, \mathtt{[]}$.

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Choose $\texttt{map} \triangleright \texttt{cons}, \texttt{[]}$.

Goal: $\texttt{map}(F, \texttt{cons}(x, y)) \succ_{\mathrm{LPO}} \texttt{cons}(F \cdot x, \texttt{map}(F, y))$

Example

$$\begin{array}{rcl} \texttt{[]} & : & o \\ \texttt{cons} & : & o \Rightarrow o \Rightarrow o \\ \texttt{map} & : & (o \Rightarrow o) \Rightarrow o \Rightarrow o \end{array}$$

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Goal: $\texttt{map}(F, \texttt{cons}(x, y)) \succ_{\mathrm{LPO}} \texttt{cons}(F \cdot x, \texttt{map}(F, y))$
Because **(greater)**:

- $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \texttt{cons}(F \cdot x, \texttt{map}(F, y))$

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Choose $\mathtt{map} \rhd \mathtt{cons}, \mathtt{[]}$.

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Because **(copy)**:

- $\mathtt{map} \rhd \mathtt{cons}$
- $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} F \cdot x$
- $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{map}(F, y))$

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Choose $\mathtt{map} \triangleright \mathtt{cons}, []$.

Goal 1: $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} F \cdot x$

Goal 2: $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{map}(F, y))$

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$$\begin{aligned} \texttt{map}(F, \texttt{[]}) &\to \texttt{[]} \\ \texttt{map}(F, \texttt{cons}(x, y)) &\to \texttt{cons}(F \cdot x, \texttt{map}(F, y)) \end{aligned}$$

Choose $\texttt{map} \rhd \texttt{cons}, \texttt{[]}$.

Goal 1: $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\text{LPO}} F \cdot x$
Goal 2: $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\text{LPO}} \texttt{map}(F, y))$
Because **(lex)**:

- $F \succeq_{\text{LPO}} F$
- $\texttt{cons}(x, y)) \succeq_{\text{LPO}} y$ (both have type o!)

Example

$$\begin{array}{rcl} \texttt{[]} & : & \mathsf{o} \\ \texttt{cons} & : & \mathsf{o} \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \texttt{map} & : & (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \end{array}$$

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Goal 2: $\texttt{cons}(x, y)) \succeq_{\mathrm{LPO}} y$
Because **(greater)**:

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Example

$$\begin{array}{rcl} \mathtt{[]} & : & o \\ \mathtt{cons} & : & o \Rightarrow o \Rightarrow o \\ \mathtt{map} & : & (o \Rightarrow o) \Rightarrow o \Rightarrow o \end{array}$$

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Goal 2: $\texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} y$

Because **(sub)**:

- $y \succeq y$

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Choose $\texttt{map} \triangleright \texttt{cons}, \texttt{[]}$.

Goal: $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\text{LPO}} F \cdot x$
Because **(app)**:

- $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\text{LPO}} F$
- $\texttt{map}(F, \texttt{cons}(x, y)) \sqsupset^{\emptyset}_{\text{LPO}} x$

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Because **(sub)**:

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Choose $\mathtt{map} \rhd \mathtt{cons}, []$.

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Because **(sub)**:

- $y \succeq_{\mathrm{LPO}} y$

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Nothing left to prove!

How to write down a HOLPO proof?

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4. $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{map}(F, y)$
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   by **(greater)**, 6
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8. $\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} x$

Exercise

Orient the following rules using HOLPO:

$$\begin{aligned} \mathtt{start} &:: \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{add} &:: \mathsf{o} \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \\ \mathtt{map} &:: (\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \Rightarrow \mathsf{o} \end{aligned}$$

$$\mathtt{start}(y) \to \mathtt{map}(\lambda x_{\mathsf{o}}.\mathtt{add}(x_{\mathsf{o}}, x_{\mathsf{o}}), y)$$

$$\begin{aligned} \mathtt{a} &:: \mathsf{o} \\ \mathtt{b} &:: \mathsf{o} \\ \mathtt{f} &:: ((\mathsf{o} \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o}) \Rightarrow \mathsf{o} \end{aligned}$$

$$\mathtt{f}(\lambda x_{\mathsf{o}\Rightarrow\mathsf{o}}.x_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot \mathtt{a}) \to \mathtt{f}(\lambda y_{\mathsf{o}\Rightarrow\mathsf{o}}.y_{\mathsf{o}\Rightarrow\mathsf{o}} \cdot \mathtt{b})$$

Polymorphic HOLPO

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Idea: be creative with the type collapsing!

Polymorphic HOLPO

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$$\texttt{collapse}(\mathsf{list}(\alpha)) := \texttt{collapse}(\alpha) \quad \text{for all types } \alpha$$

$$\begin{aligned} \texttt{cons}_1 &:: \alpha \Rightarrow \alpha \\ \texttt{cons}_2 &:: \beta \Rightarrow \beta \\ \texttt{map} &:: (\alpha \Rightarrow \beta) \Rightarrow \alpha \Rightarrow \beta \end{aligned}$$

$$\texttt{map}(F_{\alpha \Rightarrow \beta}, \texttt{cons}_1(x_\alpha, y_\alpha)) \rightarrow \texttt{cons}_2(F_{\alpha \Rightarrow \beta} \cdot x_\alpha, \texttt{map}(F_{\alpha \Rightarrow \beta}, y_\alpha))$$

HORPO

(mul) $s = \mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} \mathtt{g}(t_1, \ldots, t_n)$ if

- $\mathtt{f} \approx \mathtt{g}$
- $\mathit{status}(\mathtt{f}) = \mathit{mul}_m$ for some $m \in \mathbf{N}$ with $m \leq n$
- $\mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} t_i$ for all $i \in \{1, \ldots, n\}$
- $\{\{s_1, \ldots, s_{\mathsf{min}(k,m)}\}\} \ (\succ_{\mathrm{LPO}})_{\mathrm{mul}} \ \{\{t_1, \ldots, t_m\}\}$

Challenge: well-foundedness of HORPO

Recall: well-foundedness proof of RPO:

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if $(s_1, \ldots, s_n)$ terminating, and $\mathtt{f}(s_1, \ldots, s_n) \succ_{\text{LPO}} t$, then t terminating

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Problem: termination of, e.g., $\mathtt{map}(F, \mathtt{cons}(x, y))$ depends on the behaviour of F.

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Recall: well-foundedness proof of RPO:

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Problem: termination of, e.g., $\mathtt{map}(F, \mathtt{cons}(x, y))$ depends on the behaviour of F.

Example:

$$
\begin{aligned}
\mathtt{map}(F, []) &\rightarrow [] \\
\mathtt{map}(F, \mathtt{cons}(x, y)) &\rightarrow \mathtt{cons}(F \cdot x, \mathtt{map}(F, y)) \\
\mathtt{f}\ x &\rightarrow \mathtt{f}\ (\mathtt{s}\ x)
\end{aligned}
$$

Solution: computability

Definition

- a term s of base type is *computable* if s is terminating (under $\succ_{\text{LPO}}$)

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- a term s of type $\sigma \Rightarrow \tau$ is computable if
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(This is well-defined by induction on types.)

Properties of computability

Claim: for all types σ:

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Proof:

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Proof: by shared induction on σ

Properties of computability

Claim: for all types σ:

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Proof: by shared induction on σ (class exercise)

Soundness of HORPO

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Main proof ideas:

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- if $s[x := t]$ is computable for all computable t, then $\lambda x.s$ computable

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- if $s[x := t]$ is computable for all computable t, then $\lambda x.s$ computable
- if $s_1, \ldots, s_k$ computable, and $\mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} t$,

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- if $s_1, \ldots, s_k$ computable, and $\mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} t$, then $t[\vec{x} := \vec{u}]$ is computable for all computable $\vec{u}$

Soundness of HORPO

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- if $s[x := t]$ is computable for all computable t, then $\lambda x.s$ computable
- if $s_1, \ldots, s_k$ computable, and $\mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} t$, then $t[\vec{x} := \vec{u}]$ is computable for all computable $\vec{u}$

  (by induction first on $\mathtt{f}$,
  then on $(s_1, \ldots, s_k)$ ordered with $status(\mathtt{f})$,
  and finally on the derivation of $\mathtt{f}(s_1, \ldots, s_k) \sqsupset^X_{\mathrm{LPO}} t$)

Implementing automatic HORPO proof search

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Needed:

Implementing automatic HORPO proof search

Needed: status

Implementing automatic HORPO proof search

Needed: status, precedence

Implementing automatic HORPO proof search

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Implementing automatic HORPO proof search

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Idea:

- for each function symbol: an integer value for the precedence
- for each function symbol: an integer value for the status
- for each HORPO relation we encounter: a boolean variable

Example: encoding proof search for map

Formula:

- v_1

Variables:

- $v_1 \equiv$ "$\texttt{map}(F, \texttt{cons}(x, y)) \succ_{\text{LPO}} \texttt{cons}(F \cdot x, \texttt{map}(F, y))$"

Example: encoding proof search for map

Formula:

- v_1
- $(v_1 \rightarrow v_2 \vee v_3 \vee v_4 \vee v_5 \vee v_6)$

Variables:

- $v_1 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_2 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(sub)**"
- $v_3 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(copy)**"

Example: encoding proof search for map

Formula:

- v_1
- $(v_1 \rightarrow v_2 \vee v_3 \vee v_4 \vee v_5 \vee v_6)$
- $(v_2 \rightarrow v_7 \vee v_8)$

Variables:

- $v_1 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_2 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(sub)**"
- $v_3 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(copy)**"
- $v_7 \equiv$ "$F \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_8 \equiv$ "$\mathtt{cons}(x, y) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"

Example: encoding proof search for map

Formula:

- v_1
- $(v_1 \rightarrow v_2 \vee v_3 \vee v_4 \vee v_5 \vee v_6) \wedge$
- $(v_2 \rightarrow v_7 \vee v_8) \wedge$
- $(v_3 \rightarrow (\mathtt{prec}_{\mathtt{map}} > \mathtt{prec}_{\mathtt{cons}} \wedge v_9 \wedge v_{10})) \wedge$

Variables:

- $v_1 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_2 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(sub)**"
- $v_3 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(copy)**"
- $v_7 \equiv$ "$F \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_8 \equiv$ "$\mathtt{cons}(x, y) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_9 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} F \cdot x$"
- $v_{10} \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{map}(F, y)$"

Example: encoding proof search for map

Formula:

- v_1
- $(v_1 \rightarrow v_2 \vee v_3 \vee v_4 \vee v_5 \vee v_6) \wedge$
- $(v_2 \rightarrow v_7 \vee v_8) \wedge$
- $(v_3 \rightarrow (\mathtt{prec}_{\mathtt{map}} > \mathtt{prec}_{\mathtt{cons}} \wedge v_9 \wedge v_{10})) \wedge$
- …

Variables:

- $v_1 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_2 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(sub)**"
- $v_3 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$ **(copy)**"
- $v_7 \equiv$ "$F \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_8 \equiv$ "$\mathtt{cons}(x, y) \succ_{\mathrm{LPO}} \mathtt{cons}(F \cdot x, \mathtt{map}(F, y))$"
- $v_9 \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} F \cdot x$"
- $v_{10} \equiv$ "$\mathtt{map}(F, \mathtt{cons}(x, y)) \sqsupset^{\emptyset}_{\mathrm{LPO}} \mathtt{map}(F, y)$"