Introduction to Formal Reasoning 2019
Solutions Exam
(15/01/20)

The following exercises are concerned with propositional logic:

1. Consider the formula of propositional logic:

$$a \lor b \land c \leftrightarrow a \lor b \land a \lor c$$

(a) Write this formula according to the official grammar from the course notes.

$$\underbrace{\left(\underbrace{\left(a \lor \underbrace{(b \land c)}_{\land}\right)}_{\lor} \leftrightarrow \underbrace{\left(a \lor \underbrace{\left(\underbrace{(b \land a)}_{\land} \lor c\right)}_{\lor}\right)}_{\lor}\right)}_{\leftrightarrow}$$

Common mistakes:

- Many students used the wrong order: first $\lor$ and then $\land$.

(b) Give the truth table of this formula.

a	b	c	$b \land c$	$a \lor (b \land c)$	$b \land a$	$(b \land a) \lor c$	$a \lor ((b \land a) \lor c)$	$(a \lor (b \land c)) \leftrightarrow (a \lor ((b \land a) \lor c))$
0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	1	1	0
0	1	0	0	0	0	0	0	1
0	1	1	1	1	0	1	1	1
1	0	0	0	1	0	0	1	1
1	0	1	0	1	0	1	1	1
1	1	0	0	1	1	1	1	1
1	1	1	1	1	1	1	1	1

2. Consider the following English sentence:

I'll be damned if I don't understand this.

(a) Give a formula of propositional logic that formalizes the meaning of this sentence using the dictionary:

D	I am damned
U	I understand this

$$\neg U \to D$$

(b) Explain why this sentence expresses that I understand this.

The basic idea of the English expression is that 'I am not damned'. So if $\neg U \rightarrow D$ is true, and D is false, then $\neg U$ must also be false, implying that U is true. So 'I understand this'.

3. Consider the model v_3 of propositional logic with $v_3(a) = 1$ and $v_3(b) = 0$. Explain what corresponds in a truth table with this model.

   The row that has a 1 in the column for a and a 0 in the column for b.

The following exercises are concerned with predicate logic:

4. Consider the following English sentence:

   Nobody likes everything.

   Give a formula of predicate logic that formalizes the meaning of this sentence using the dictionary:

H	the domain of people
T	the domain of things
$L(x, y)$	x likes y

The sentence

Nobody likes everything.

can be reformulated as

There isn't a person who likes all things.

This leads to the formula:

$$\neg\,(\exists h \in H\ \ (\forall t \in T\, L(h, t)))$$

5. Consider the following English sentence:

   There are exactly three countries in the world not using the metric system.

   Give a formula of predicate logic with equality that formalizes the meaning of this sentence using the dictionary:

C	the domain of countries in the world
$M(x)$	x uses the metric system

First a solution using step-by-step introduction of variables:

$$\begin{array}{llllllll}
\exists x \in C & [& \neg M(x) \wedge & & & & & \\
& & \exists y \in C & [& \neg(x = y) \wedge \neg M(y) \wedge & & & \\
& & & & \exists z \in C & [& \neg(x = z) \wedge \neg(y = z) \wedge \neg M(z) \wedge & \\
& & & & & & \forall u \in C & [\neg M(u) \rightarrow u = x \vee u = y \vee u = z] \\
& & & & &] & & \\
& & &] & & & & \\
&] & & & & & &
\end{array}$$

And here a solution using some shorthand notation:

$$\exists x, y, z \in C\, [\neg M(x) \wedge \neg M(y) \wedge \neg M(z) \wedge x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u \in C\, [\neg M(u) \rightarrow u = x \vee u = y \vee u = z]]$$

6. We are looking for a model of predicate logic, and an interpretation in this model that makes the following formula of predicate logic with equality true:

$$\exists y_1, y_2 \in X\,[y_1 \neq y_2] \wedge \forall x \in X\, \exists y_1, y_2 \in X\,[x \neq y_1 \wedge x \neq y_2]$$

Give such a model and interpretation where the interpretation of X has a *minimal* number of elements. Explain your answer.

Though the formula seems to suggest that set X contains at least three elements, it actually only states that set X contains at least two elements, because there is no $y_1 \neq y_2$ in the last part of the formula.

We take the following trivial model:

Domain(s)	{Freek, Engelbert}
Predicate(s)	(none)
Relation(s)	(none)

And this also trivial interpretation:

X	{Freek, Engelbert}

The formula holds because:

- For the first y_1 we can take Freek, and for the first y_2 we can take Engelbert. Then clearly $y_1 \neq y2$.
- If the x is taken to be Freek, then we can take both the second y_1 and y_2 to be Engelbert and we get Freek $\neq$ Engelbert $\wedge$ Freek $\neq$ Engelbert, which clearly holds.
- If the x is taken to be Engelbert, then we can take both the second y_1 and y_2 to be Freek and we get Engelbert $\neq$ Freek $\wedge$ Engelbert $\neq$ Freek, which clearly holds.

So we have shown that both components of the conjunction hold, hence the conjunction itself holds.

Note: Any model with a domain with exactly two elements will do. No further predicates or relations are needed, because the default equality suffices. For domains with less elements, the first part of the conjunction doesn't hold. For domains with three or more elements, the formula holds, but the model is not minimal as requested.

The following exercises are concerned with formal languages:

7. Give an infinite language L_7 (i.e., that contains infinitely many different words), such that

$$L_7{}^R \subseteq \overline{L_7}$$

As a reminder: the reverse of a language is defined by:

$$L^R := \{w^R \mid w \in L\}$$

Explain your answer.

So what we are looking for is a language for which none of the words, when reversed, are part of the original language. Take the language

$$L_7 := \{a^n b^n \mid n \in \mathbb{N} \text{ and } n \geq 1\}$$

This language complies to all requirements:

- It is an infinite language because there are infinitely many choices for $n \in \mathbb{N}$ such that $n \geq 1$.
- In addition ${L_7}^R := \{b^n a^n \mid n \in \mathbb{N} \text{ and } n \geq 1\}$.
- And $\overline{L_7} = \{w \in \{a,b\}^* \mid \text{there is no } n \in \mathbb{N} \text{ such that } n \geq 1 \text{ and } w = a^n b^n\}$.
- And obviously, if $w \in {L_7}^R$ then w starts with a b and hence there is no $n \in \mathbb{N}$ such that $n \geq 1$ and $w = a^n b^n$. Hence $w \in \overline{L_7}$.

Note that the requirement $n \geq 1$ is needed to prevent that $\lambda \in L_7$, which would imply that $\lambda \in {L_7}^R$, but clearly $\lambda \notin \overline{L_7}$.

Common mistakes:

- Many students had given a language that contains λ, a or b. In this case, there are words in L_7 that are equal to its reverse and therefore in not in its complement;
- Many students had argued that the given language did not contain palindromes. Note that the condition does not ask for this;
- Many students had given a language that was equal to its reverse.

8. Give a regular expression for the language:

$$L_8 := \{w \in \{a,b,c\}^* \mid w \text{ both contains } ab \text{ and } bc\}$$

These are the basic options for a word $w \in L_8$:

- w contains ab and later on it contains bc;
- w contains abc;
- w contains bc and later on it contains ab.

This leads to the regular expression:

$$\begin{array}{ll} & ((a \cup b \cup c)^* ab (a \cup b \cup c)^* bc (a \cup b \cup c)^*) \\ \cup & ((a \cup b \cup c)^* abc (a \cup b \cup c)^*) \\ \cup & ((a \cup b \cup c)^* bc (a \cup b \cup c)^* ab (a \cup b \cup c)^*) \end{array}$$

Common mistakes:

- Many students missed that ab and bc can overlap, so that the regular expression also has to match the string abc.
- Several students missed that ab and bc do not have to be in that order, so that the regular expression also has to match the string $bcab$.
- Most students used the standard idiom $(a \cup b \cup c)^*$ for 'any string', but some used the also correct, but strange idiom $(a^*b^*c^*)^*$. However some students forgot the outer star in this and used the incorrect $a^*b^*c^*$.
- Some student did not realize that a and A are different letters in formal reasoning, and gave a regular expression containing uppercase letters.

9. Let be given the context-free grammar G_9:

$$S \to aSb \mid \lambda$$

Show that the following property is an invariant of this grammar:

$$P(w) := w \text{ contains the same number of } a\text{'s and } b\text{'s}$$

So we have to show two things:

(a) $P(S)$ holds. This is obviously true since S contains zero a's and zero b's, so in particular S contains the same amount of a's and b's.

(b) Let v be a word such that $P(v)$ holds and let v' be a word such that $v \to v'$. We now have to show that $P(v')$ also holds. There are two rules that can have caused $v \to v'$:

- $S \to aSb$. In this case v' contains exactly one a more than v and also one b more. But since the amount of a's and b's was the same in v, it is also the same in v'.
- $S \to \lambda$. In this case the amount of a's and b's in v' is the same as in v, and v contained the same amount of a's and b's, so v' also contains the same amount of a's and b's.

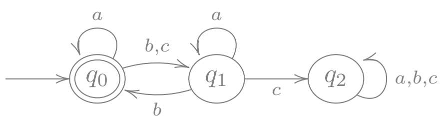
So the property $P(w)$ is indeed an invariant.

Common mistakes:

- Many students did not realize that the property of being an invariant has *two* requirements, and missed the fact that they had to show that $P(S)$ holds. As this counted for two out of the five points, that was a serious omission.
- Students that did remember that they had to check that $P(S)$ holds, clearly did not understand what this means. They were considering the right hand sides of the rules for S, that is: aSb and λ. But this has nothing to do with the fact that P holds for the one symbol string 'S'. The correct argument is that this one symbol string contains zero a's and b's and that zero is equal to zero.
- Many students missed that for an invariant they need to check that the property is invariant for *any* string, and not just for strings that can be produced by the grammar. In this case the strings that can be produced have the form a^nSb^n and a^nb^n, but the productions $v \to v'$ that have to be considered also have to include v's that are not of this form.

The following exercises are concerned with automata:

10. Give the right linear context-free grammar that corresponds to the deterministic finite automaton M_{10}:



Make sure your grammar also has a non-terminal that corresponds to state q_2.

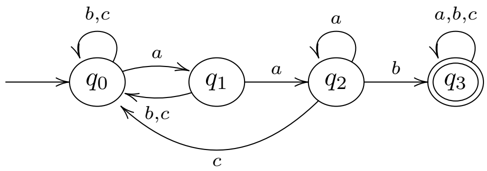
We identify state q_0 by non-terminal S, q_1 by A and q_2 by B.

$$\begin{array}{lcl} S & \to & aS \mid bA \mid cA \mid \lambda \\ A & \to & aA \mid bS \mid cB \\ B & \to & aB \mid bB \mid cB \end{array}$$

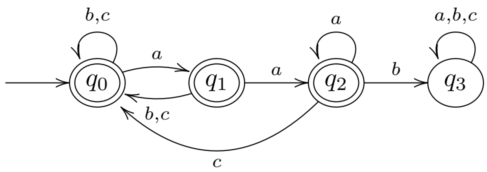
11. Give a deterministic finite automaton that recognizes the language:

$$L_{11} := \{w \in \{a, b, c\}^* \mid w \text{ does not contain } aab\}$$

We start by creating an automaton that accepts only the words containing aab.



And then we swap final states with non-final states:



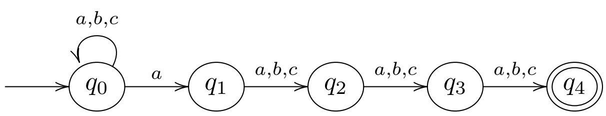
Common mistakes:

- A surprisingly frequent mistake was that the a transition from q_2 did not loop back to q_2, but instead 'restarted' the machine by going back to state q_0. In that case the word $aaab$ is accepted, which is not in the language, as it *does* contain aab.
- Some people missed that the alphabet included the symbol c, and gave an automaton that had only a and b transitions.

12. Give a non-deterministic finite automaton with at most five states that recognizes the language:

$$L_{12} := \mathcal{L}((a \cup b \cup c)^* a(a \cup b \cup c)(a \cup b \cup c)(a \cup b \cup c))$$

So this language accepts all words that have at least length four and where the fourth character from the end has to be an a. Hence any word starts with some repetitions of a, b or c, followed by the obligatory a, followed by the obligatory remaining three letters:



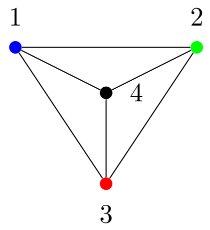
Common mistakes:

- The only recurring mistake was that some students did not just use a loop at the start of the machine (sometimes also involving a lambda transition), and therefore used six instead of the required maximum of five states.
- Also a few students forgot this loop at the start of the machine altogether.
- Another mistake that occured a few times was that the last state of the machine was not marked as an accepting state.

The following exercises are concerned with discrete mathematics:

13. A graph is called *cubic* if every vertex has degree three. Give a connected, bridgeless, planar, cubic graph that contains a Hamiltonian cycle. Explain why your graph has all five requested properties.

$G_{13} = K_4$



(a) The graph is connected. From every vertex there is a path to every other vertex. (Even an edge.) It consists of one component.

(b) The graph is bridgeless: If we remove any of the edges, the graph still consists out of one component.

(c) The graph is planar. It has been drawn without any crossing edges.

(d) The graph is cubic. Each vertex has three neighbors: 1 has 2, 3 and 4; 2 has 1, 2 and 3; 3 has 1, 2 and 4; 4 has 1, 2 and 3.

(e) The graph has a Hamiltonian cycle. The cycle $1 \to 2 \to 3 \to 4 \to 1$ visits each vertex exactly once.

Common mistakes:

- Many students did not know what a bridge was, so they did not do and/or remember exercise 5.B from the course notes. Several incorrectly thought that a bridge is an edge in a non-planar graph that crosses another edge.
- Many students did not use the formal definition of connectedness in terms of paths. Also, many students claimed a graph is connected if there are no vertices of degree zero (isolated points). This is not correct: a graph with two components, where each component is isomorphic to the K_2, has this property, but is not connected.
- Several students used a non-standard notation for the Hamilton cycle, and/or forgot the final step back to the starting vertex.
- Almost all students had a correct graph with all five required properties (that already gave them 2.5 out of the 5 points, and as we rounded up, they already had 3 points that way). The only exceptions gave graphs with multiple edges between two points. This is not allowed in graphs as they are defined in this course.
- One student noted that there is redundancy in the five requirements: a graph with a Hamiltonian cycle is always connected,

14. We define a sequence $(a_n)_{n\geq 3}$ using the recursive equations:

$$a_3 = 4$$
$$a_{n+1} = a_n + 2n + 1 \quad \text{for } n \geq 3$$

(a) Show how to use this recursive definition to compute a_5.

$$\begin{array}{lllllllll} a_4 & = & a_{3+1} & = & a_3 + 2\cdot 3 + 1 & = & 4+6+1 & = & 11 \\ a_5 & = & a_{4+1} & = & a_4 + 2\cdot 4 + 1 & = & 11+8+1 & = & 20 \end{array}$$

Common mistakes:

- Wrong interpretation of n: $a_4 = a_{4+1}$.
- Wrong substitution of n: $a_4 = a_{3+1} = a_3 + 2 \cdot \boxed{4} + 1$

(b) Prove by induction that $a_n = n^2 - 5$ for all $n \geq 3$.

Proposition: 0

$a_n = n^2 - 5$ for all $n \geq 3$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad a_n = n^2 - 5$$ 2

Base Case. We show that $P(3)$ holds, i.e. we show that 3

$a_3 = 3^2 - 5$

This indeed holds, because 4

$a_3 = 4 = 9 - 5 = 3^2 - 5$.

Induction Step. Let k be any natural number such that $k \geq 3$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = k^2 - 5$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = (k+1)^2 - 5$

This indeed holds, because 8

$$\begin{aligned} a_{k+1} &= a_k + 2 \cdot k + 1 && \text{definition of } a_n \\ &= k^2 - 5 + 2 \cdot k + 1 && \text{IH} \\ &= k^2 + 2 \cdot k + 1 - 5 && \text{elementary algebra} \\ &= (k+1)^2 - 5 && \text{elementary algebra} \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 3$. 9

Common mistakes:

- No definition of $P(n)$.
- Wrong definition of $P(n) := n^2 - 5$.
- Wrong explanation what should be done in the induction step:
  - We assume $P(k)$ holds for every k.
  - We assume that $P(k+1)$ holds.

(c) Use the previous sub-exercise to compute a_{45}.

From the previous sub-exercise it follows that $a_{45} = 45^2 - 5 = 2020$, because

$$\begin{array}{r} 4\,5 \\ \times\ 4\,5 \\ \hline 2\,2\,5 \\ 1\,8\,0\,0 \\ \hline 2\,0\,2\,5 \end{array} \quad \text{and} \quad \begin{array}{r} 2\,0\,2\,5 \\ -\quad\ \ 5 \\ \hline 2\,0\,2\,0 \end{array}$$

Common mistakes:

- Wrong squaring $45^2 = (40+5)^2 = 1600 + 25 = 1625$ instead of $45^2 = (40+5)^2 = (40^2 + 2 \cdot 40 \cdot 5 + 5^2) = 1600 + 400 + 25 = 2025$.

- Usage of the recursive definition instead of the direct formula.

15. We want to put four objects in two boxes, where boxes may be left empty. We will count the number of ways that this can be done in four different variants.

    In each variant giving the number of ways is sufficient, you do not need to explain how you obtained your answer. This means also that you do not need to relate your answers to binomial coefficients, Stirling numbers or Bell numbers.

    However, if you are insecure about whether you got the correct answer, explanations are allowed.

    (a) In how many ways can one put four objects in two boxes, where both the objects and the boxes are indistinguishable.

    So the only thing that matters is the amount of objects in each box. This can be done in **three** ways: $4-0$, $3-1$ or $2-2$, where $x-y$ means x objects in the first box and y objects in the second box. Note that we don't need $1-3$ and $0-4$, because these are the same as $3-1$ and $4-0$, because the boxes are indistinguishable.

    (b) In how many ways can one put four objects in two boxes, where the objects are indistinguishable but the boxes are distinguishable.

    In this case we get **five** possibilities. Again, only the amount of objects in each box is important, but this time $1-3$ and $3-1$ are actually different distributions. So the options are $4-0$, $3-1$, $2-2$, $1-3$ or $0-4$.

    Common mistakes:

    - The answer of the previous question is doubled.

    (c) In how many ways can one put four objects in two boxes, where the objects are distinguishable but the boxes are indistinguishable.

    This can be done in **eight** ways. The Stirling number of the second kind tells us that there are $\left\{ {4 \atop 2} \right\} = 7$ ways to distribute four distinguishable objects over two non-empty indistinguishable boxes, but since it is allowed to have empty boxes, we must add 1 for the option that one of the two boxes is empty. These are the options, if we assume that the distinguishable objects are the numbers 1, 2, 3 and 4:

$$\begin{gathered}\{1,2,3,4\},\{\}\\ \{1,2,3\},\{4\}\\ \{1,2,4\},\{3\}\\ \{1,3,4\},\{2\}\\ \{2,3,4\},\{1\}\\ \{1,2\},\{3,4\}\\ \{1,3\},\{2,4\}\\ \{1,4\},\{2,3\}\end{gathered}$$

    Common mistakes:

    - Only $\left\{ {4 \atop 2} \right\}$ is given and the special case where everything is in one box $\left\{ {4 \atop 1} \right\}$ is forgotten.

(d) In how many ways can one put four objects in two boxes, where both the objects and the boxes are distinguishable.

This can be done in **sixteen** ways. First we distribute the four distinguishable objects into two indistinguishable boxes, which can be done in eight ways as we have seen above. And then we label the boxes, which can be done in two ways. So in the end these are the sixteen distributions that we get:

$\{1,2,3,4\}, \{\}$	$\{\}, \{1,2,3,4\}$
$\{1,2,3\}, \{4\}$	$\{4\}, \{1,2,3\}$
$\{1,2,4\}, \{3\}$	$\{3\}, \{1,2,4\}$
$\{1,3,4\}, \{2\}$	$\{2\}, \{1,3,4\}$
$\{2,3,4\}, \{1\}$	$\{1\}, \{2,3,4\}$
$\{1,2\}, \{3,4\}$	$\{3,4\}, \{1,2\}$
$\{1,3\}, \{2,4\}$	$\{2,4\}, \{1,3\}$
$\{1,4\}, \{2,3\}$	$\{2,3\}, \{1,4\}$

The following exercises are concerned with modal logic:

16. (a) Explain the difference between epistemic logic and doxastic logic.

Epistemic logic is about knowledge, and doxastic logic is about belief. In particular $\Box f$ means *I know that f holds* in epistemic logic, and it means *I believe that f holds* in doxastic logic.

Common mistakes:

- Swapping epistemic and doxastic logic.
- Interpreting doxastic logic as deontic logic.

(b) Give an axiom scheme that holds in one of these logics, but not in the other. Explain your answer. *Note:* An axiom scheme that holds is a formula that holds independent from the interpretation of the propositional variables. In particular, you don't have to know the name of your axiom scheme.

Take the axiom scheme for reflexivity: $\Box f \to f$. It holds in epistemic logic: if one knows that f holds, then f really holds. But it doesn't hold in doxastic logic: if one believes that f holds, then that doesn't mean that f actually holds; many people believe things that are not true. *Common mistakes:*

- No explanation given.
- A conjunction like $\Box f \wedge f$.
- Formulas that are clearly not true *for all interpretations* of f and/or g like $\Box f \to \Diamond\Box g$.
- Schemes holding in both systems: $\neg\Diamond f \to \Box\neg f \equiv \Box\neg f \to \Box\neg f$, $\Box f \to \Box\Box f$.
- Schemes holding in neither of the systems: $\Diamond f \to \Box f$, $\Box f \to \neg\Diamond f \equiv \Box f \to \Box\neg f$, $\Box\Box f$.

17. In the semantics of modal logic we use the symbols '$\Vdash$' and '$\models$'. Below there are eight different notations using these symbols. Write down in which case(s) the symbol is used correctly. *Note:* You don't have to explain what these notations mean.

$$\begin{array}{cc} \mathcal{M}, x_1 \Vdash f & \mathcal{M}, x_1 \vDash f \\ x_1 \Vdash f & x_1 \vDash f \\ \mathcal{M} \Vdash f & \mathcal{M} \vDash f \\ \Vdash f & \vDash f \end{array}$$

These four are used correctly:

- $\mathcal{M}, x_1 \Vdash f$: f holds in world x_1 in Kripke model $\mathcal{M}$.
- $x_1 \Vdash f$: f holds in world x_1 in a Kripke model which is clear from the context.
- $\mathcal{M} \vDash f$: f holds in all worlds of the Kripke model $\mathcal{M}$.
- $\vDash f$: f holds in all worlds of all Kripke models.

Common mistakes:

- Often for the statements about worlds, the wrong $\vDash$ was chosen.

18. Consider the following English sentence:

    I will only worry about Formal Reasoning until I pass the exam.

    Give an LTL formula that formalizes the meaning of this sentence using the dictionary:

W	I worry about Formal Reasoning
E	I pass the exam of Formal Reasoning

    You may assume that the atomic formula E is true at exactly one moment in time.

    Hint: The sentence does not imply that before passing the exam you are constantly worrying (maybe at some times you are not thinking about it), just that you will stop worrying once you pass the exam.

    So at any moment in time it holds that if I pass the exam at that moment, from that moment on I will never worry again about Formal Reasoning. This can be represented by this formula:

    $$\mathcal{G}(E \to \mathcal{G}(\neg W))$$

    Common mistakes:

    - Although the hint is quite explicit that the only thing you really know is that after passing the exam, you will never worry, many people tried to do something with an $f\mathcal{U}g$. However, such an $f\mathcal{U}g$ basically only states something about f holding before g holds, and nothing about what happens after g is true.
    - Furthermore, it is explicitly stated that before the exam is passed you don't worry all the time. SO it is really unclear what happens before this moment.
    - The $f\mathcal{R}g$ has a similar problem: it also doesn't state what happens after f is true.
    - In addition many students tried to release the wrong thing as in $W\mathcal{R}E$ as opposed to $E\mathcal{R}W$, which is still wrong for the previously mentioned reason, but better.
    - Many students used the general notations $\Box$ and $\Diamond$ instead of the specific LTL-notations $\mathcal{G}$ and $\mathcal{F}$.