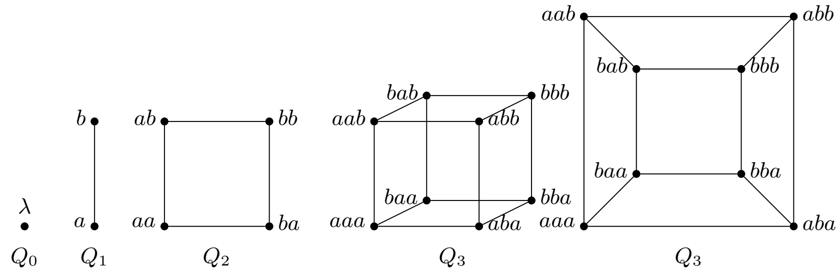
Formal Reasoning 2019
Solutions Test Block 3: Discrete Mathematics and Modal Logic
(18/12/19)

1. The *hypercube graph* Q_n has as its vertices the elements of $\{a, b\}^n$ (the words of length n over the alphabet $\{a, b\}$). Two vertices are connected by an edge when they differ in exactly one position. For example in Q_4 there is an edge between $aaab$ and $abab$, but not between $aaab$ and $abbb$. (15 points)

   Draw Q_0, Q_1, Q_2 and Q_3.



   Common mistakes:

   - Many students got Q_0 wrong. The nodes represent all words of length zero, which means that there is one node with label λ.
   - Some students did not list all words of length three systematically and forgot one or two words.
   - Some students had no idea what was asked; they often created a grap with all Q_i's as nodes.

2. In Pascal's triangle, we have:

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

   (a) Give the values of these three binomial coefficients, and show that the equation holds. (10 points)

   Note that
$$\binom{5}{3} = 10 = 6 + 4 = \binom{4}{2} + \binom{4}{3}$$

   Common mistakes:

   - Students forgot to start counting at 0 instead of 1 in Pascal's triangle, resulting in $\binom{4}{2} = 4$ and $\binom{4}{3} = 6$.
   - In the same vein, students gave values for $\binom{4}{2}$, $\binom{3}{1}$ and $\binom{3}{2}$.

   (b) Give the list of selections from the set $\{1, 2, 3, 4, 5\}$ that corresponds to the first binomial coefficient, and divide that list in two sublists that correspond to the two other binomial coefficients. Explain how this division shows why this equation holds. (15 points)

$\binom{5}{3}$ corresponds to the number of ways we can take three elements of the set $\{1, 2, 3, 4, 5\}$. These are all the options:

$$\begin{array}{c} \{1,2,3\} \\ \{1,2,4\} \\ \{1,2,5\} \\ \{1,3,4\} \\ \{1,3,5\} \\ \{1,4,5\} \\ \{2,3,4\} \\ \{2,3,5\} \\ \{2,4,5\} \\ \{3,4,5\} \end{array} = \begin{array}{c} \{1,2,5\} \\ \{1,3,5\} \\ \{1,4,5\} \\ \{2,3,5\} \\ \{2,4,5\} \\ \{3,4,5\} \end{array} + \begin{array}{c} \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \end{array}$$

The division is based on the fact whether 5 is part of the subset or not.

- All sets of three elements that include 5, can be created by selecting two elements of the set $\{1, 2, 3, 4\}$, which can be done in $\binom{4}{2}$ ways and adding the element 5 to these subsets.
- All sets of three elements that do not include 5, can be created by selecting three elements of the set $\{1, 2, 3, 4\}$, which can be done in $\binom{4}{3}$ ways.

Common mistakes:

- Students often drew partitions.
- Others gave correct sets and divisions, but failed to explain how they relate to $\binom{4}{2}$ and $\binom{4}{3}$.
- Some gave the sets associated with $\binom{4}{2}$, but failed to indicate with which sets associated with $\binom{5}{3}$ they correspond.

3. We use the dictionary:

R	it rains
W	I am wet

(a) Using a *temporal* interpretation for the modalities (i.e., interpreting '$\Box$' as 'always'), explain the difference in meaning between the two modal formulas: (15 points)

$$\Box(R \to W)$$
$$R \to \Box W$$

The first sentence means literally:

It always holds, that if it rains I am wet.

The second sentence means literally:

If it rains now, I always am wet.

However, in temporal logic 'always' means 'now and ever after'. So to stress the difference between the sentences, we can translate them as:

It holds now and ever after, that if it rains I am wet.

The second sentence means literally:

If it rains now, I am wet, now and ever after.

Common mistakes:

- Some students thought that a difference between the two formulas was that in one case the rain had to be the cause of getting wet, while in the other case the wetness could also be caused by something else. This is not part of these formulas: in both cases there is no causal relation between the rain and the wetness indicated. There just is a temporal relation.
- Some students translated $R \to W$ as 'it rains and therefore I get wet', which means that this formula implies that it actually rains. But that is the formula $R \wedge W$ (or the logically equivalent $R \wedge (R \to W)$), instead of $R \to W$.
- Some students interpreted the formula $\Box(R \to W)$ as $(\Box R) \to (\Box W)$. These are different formulas, and although the second is a logical consequence of the first (this is an instance of axiom scheme K, which holds in all Kripke models), they are not logically equivalent. In particular $\Box(R \to W)$ is not specifically about the situation in which it always rains.
- Some students gave the meaning of the formula by saying the formula *implied* its meaning. It is better to say that that is what it *means.*

(b) Which of these two formulas most accurately gives the meaning of the sentence: (10 points)

When it rains, I always get wet.

Explain your answer.

It is the first one. The sentence

When it rains, I always get wet.

contains a *when* and not an *if.* This means that the sentence can or should be read as

Whenever it rains, I get wet.

This clearly matches the first option better.

Common mistakes:

The most frequent mistake was to blindly translate the structure of the English sentence without looking at the *meaning* of the sentence. Because in the English sentence the 'always' only seems to refer to the being wet, many people chose the second formula, in which the box only refers to the atomic formula W. However, because this literally means that once it rains, you will be wet forever, this is the wrong choice. We intentionally used 'when' and 'get' to indicate that the getting wet only applied at the moment of the rain, and not after the rain stopped again. And this is the meaning of the first formula.

4. We define a sequence (a_n) using the recursion equations:

$$\begin{aligned} a_0 &= 0 \\ a_{n+1} &= a_n + 9 \cdot 10^n \qquad \text{for } n \geq 0 \end{aligned}$$

(a) Show how to compute a_3 from this definition. (10 points)

$$\begin{array}{lllllllll} a_0 & = & 0 & & & & & & \\ a_1 & = & a_{0+1} & = & a_0 + 9 \cdot 10^0 & = & 0 + 9 \cdot 1 & = & 9 \\ a_2 & = & a_{1+1} & = & a_1 + 9 \cdot 10^1 & = & 9 + 9 \cdot 10 & = & 99 \\ a_3 & = & a_{2+1} & = & a_2 + 9 \cdot 10^2 & = & 99 + 9 \cdot 100 & = & 999 \end{array}$$

Common mistakes:

- Some students got the n wrong: instead of reading a_3 as a_{2+1} implying $n = 2$, they took $n = 3$.
- Some students read $\cdot$ instead of the first $+$.
- Some students thought that $10^0 = 0$.

(b) Prove using induction that $a_n = 10^n - 1$ for all $n \geq 0$. (15 points)

Proposition: 0

$a_n = 10^n - 1$ for all $n \geq 0$.

Proof by induction on n. 1

We first define our predicate P as:

$$P(n) \quad := \quad a_n = 10^n - 1$$ 2

Base Case. We show that $P(0)$ holds, i.e. we show that 3

$a_0 = 10^0 - 1$

This indeed holds, because 4

$a_0 = 0 = 1 - 1 = 10^0 - 1$.

Induction Step. Let k be any natural number such that $k \geq 0$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$a_k = 10^k - 1$ (Induction Hypothesis IH)

We now show that $P(k+1)$ also holds, i.e. we show that 7

$a_{k+1} = 10^{k+1} - 1$

This indeed holds, because 8

$$\begin{array}{llll} a_{k+1} & = & a_k + 9 \cdot 10^k & \text{definition of } a_n \\ & = & 10^k - 1 + 9 \cdot 10^k & \text{IH} \\ & = & 10 \cdot 10^k - 1 & \text{elementary algebra} \\ & = & 10^{k+1} - 1 & \text{elementary algebra} \end{array}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$. 9

Common mistakes:

- Most important mistake was not or incorrectly defining $P(n)$. Very often $P(n)$ is defined as the **number** $10^n - 1$ instead of the **boolean** $a_n = 10^n - 1$.
- In the Base Case very often it is only shown that $10^0 - 1 = 0$, but it is not shown that $a_0 = 0$, which holds by definition. The reference to this definition should be explicitly given.
- There is no distinction between step 7 (what to prove) and step 8 (the actual proof).