Formal Reasoning 2019
Solutions Test Blocks 1, 2 and 3: Additional Test
(08/01/20)

1. We can translate LTL formulas into predicate logic with equality using the following dictionary: (30 points)

T	domain of time instances
F	domain of atomic propositions
t_0	the time instance of the world x_0
a	atomic proposition a
b	atomic proposition b
$B(i,j)$	time i is strictly before time j
$X(i,j)$	time j is one step after time i
$H(i,f)$	atomic proposition f is true at time i

For example $x_1 \Vdash \mathcal{G}a$ translates to:

$$\exists t_1 \in T\, [X(t_0,t_1) \land \forall i \in T\, [B(t_1,i) \lor t_1 = i \to H(i,a)]]$$

Give a translation in this style of:

$$x_1 \Vdash a\,\mathcal{U}\,b$$

This is the definition from the course notes:

$$\begin{array}{ll} x_i \Vdash f\,\mathcal{U}\,g & \text{there is a } j \geq i \text{ such that } x_j \Vdash g \text{ and for all} \\ & k \in \{i, i+1, \ldots, j-1\} \text{ we have } x_k \Vdash f \end{array}$$

Translating this to the formula at hand we get:

$$\begin{array}{ll} x_1 \Vdash a\,\mathcal{U}\,b & \text{there is a } j \geq 1 \text{ such that } x_j \Vdash b \text{ and for all} \\ & k \in \{1, 1+1, \ldots, j-1\} \text{ we have } x_k \Vdash a \end{array}$$

If we follow the structure from the example, we get:

$$\begin{array}{lllll} \exists t_1 \in T & [& X(t_0,t_1) \land & & \\ & & \exists j \in T & [& (B(t_1,j) \lor t_1 = j) \land H(j,b) \land \\ & & & & \forall k \in T\, [(B(t_1,k) \lor t_1 = k) \land B(k,j) \to H(k,a)] \\ & & &] & \\ &] & & & \end{array}$$

However, $j \geq 1$ is equivalent to $j > 0$, so we can also try to give a formula for:

$$\begin{array}{ll} x_1 \Vdash a\,\mathcal{U}\,b & \text{there is a } j > 0 \text{ such that } x_j \Vdash b \text{ and for all} \\ & k \in \{1, 1+1, \ldots, j-1\} \text{ we have } x_k \Vdash a \end{array}$$

And because of the fact that $B(i,j)$ indicates i is strictly before j, this simplifies the formula, because we no longer need t_1.

$$\begin{array}{lll} \exists j \in T & [& B(t_0,j) \land H(j,b) \land \\ & & \forall k \in T\, [B(t_0,k) \land B(k,j) \to H(k,a)] \\ &] & \end{array}$$

2. Consider the context-free grammar G_2 : (30 points)

$$S \to A \mid bS$$
$$A \to aA \mid cS \mid \lambda$$

We can show that $ab \notin \mathcal{L}(G_2)$ by using the invariant:

$$P(w) := \big(w \text{ does not contain any of: } ab,\ aS,\ Ab,\ Sb,\ AS,\ SS\big)$$

To show that this is an invariant, you need to check that $P(S)$ holds (that is easy), but you also need to check that the property is preserved by each production rule in the grammar. The exercise is to show that this is the case for the rule:

$$S \to bS$$

(You do not need to show this for the other four rules.)

So what we have to show is that given any word $v \in \{a, b, S, A\}^*$ such that $P(v)$ holds, and any word $v' \in \{a, b, S, A\}^*$ such that v' is produced from v by the production $S \to bS$, $P(v')$ also holds.

Now let us assume that v and v' are as above. Then in particular v should contain an S. So we know that $v = uSw$ for some $u, w \in \{a, b, S, A\}^*$.

Then it follows that $v' = ubSw$.

Because we know that $P(v)$ holds, it is clear that if $P(v')$ doesn't hold, it must have been caused by the surroundings of the bS in v'. In fact there are five ways that $P(v')$ may not hold:

- If $u = u'a$ then $v' = u'abSw$ and $P(v')$ doesn't hold because v' contains ab. But this cannot happen, because it would imply that $v = u'aSw$ which contains the forbidden aS, so $P(v)$ wouldn't hold.
- If $u = u'A$ then $v' = u'AbSw$ and $P(v')$ doesn't hold because v' contains Ab. But this cannot happen, because it would imply that $v = u'ASw$ which contains the forbidden AS, so $P(v)$ wouldn't hold.
- If $u = u'S$ then $v' = u'SbSw$ and $P(v')$ doesn't hold because v' contains Sb. But this cannot happen, because it would imply that $v = u'SSw$ which contains the forbidden SS, so $P(v)$ wouldn't hold.
- If $w = bw'$ then $v' = ubSbw'$ and $P(v')$ doesn't hold because v' contains Sb. But this cannot happen, because it would imply that $v = uSbw$ which contains the forbidden bS, so $P(v)$ wouldn't hold.
- If $w = Sw'$ then $v' = ubSSw'$ and $P(v')$ doesn't hold because v' contains SS. But this cannot happen, because it would imply that $v = uSSw$ which contains the forbidden SS, so $P(v)$ wouldn't hold.

Hence $P(v')$ holds as well, so the property $P(w)$ is invariant under the production $S \to bS$.

3. Consider the following equation:

$$\left\{ {n \atop n-2} \right\} = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$$

(a) Show that this equation holds for $n = 4$ by computing the Stirling number and the binomial coefficients explicitly. (10 points)

$$\left\{ {4 \atop 4-2} \right\} = 7 \qquad \binom{4}{3} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4-2}{2} = 1$$

And $7 = 4 + \frac{1}{2} \cdot 6 \cdot 1$.

(b) Give a combinatorial argument that shows that this equation holds for all $n \geq 4$. (20 points)

Note that by definition $\left\{ {n \atop n-2} \right\}$ is the number of ways we can distribute n distinct objects over $n-2$ indistinguishable bags, where the bags should be non-empty.

Because the bags are supposed to be non-empty, there isn't much choice. Any distribution can be created by first putting $n-2$ objects into a bag on their own. Then we have two objects left. There are two possibilities:

- Either these last two objects are placed together in one of the $n-2$ bags.
- Or these last two objects are placed in two different bags. Note that this works because $n \geq 4$ and hence $n-2 \geq 2$.

A distribution of the first kind can be created in the following way:

- First we choose three objects and put them together in a single bag. This can be done in $\binom{n}{3}$ ways.
- Then we put each of the remaining $n-3$ objects in of the remaining $n-3$ bags on their own. This can be done in only one way, since the bags are indistinguishable.

Hence there are $\binom{n}{3}$ distributions of the first type.

A distribution of the second kind can be created in the following way:

- First we choose two objects and put them together in a single bag. This can be done in $\binom{n}{2}$ ways.
- Then we choose two objects out of the remaining $n-2$ objects and put them together in a single bag (obviously in one of the remaining $n-1$ bags). This can be done in $\binom{n-2}{2}$ ways.
- Then we put each of the remaining $n-4$ objects in of the remaining $n-4$ bags on their own. This can be done in only one way, since the bags are indistinguishable.

So this would give $\binom{n}{2} \cdot \binom{n-2}{2}$ distributions, however, we have counted everything double. Because the bags are indistinguishable it doesn't matter which two objects we put in the first bag and which two in the second bag. So we have to divide by two. Hence there are $\frac{1}{2} \cdot \binom{n}{2} \cdot \binom{n-2}{2}$ distributions of the second type.

Hence in total there are

$$\binom{n}{3} + \frac{1}{2} \cdot \binom{n}{2} \cdot \binom{n-2}{2}$$

ways to distribute n distinct objects over $n-2$ indistinguishable non-empty bags.