Formal Reasoning 2020
Exam
(14/01/21)

There are six sections, with each three multiple choice questions and one open question. Each multiple choice question is worth 3 points, and the open questions are worth 6 points. The first ten points are free. Good luck!

Propositional logic: multiple choice questions

1. We use the dictionary:

W	it is winter
S	it snows

   What is the best formalization of the sentence:

   It only snows in winter.

   (a) $W \to S$
   (b) $S \to W$
   (c) $S \leftrightarrow W$
   (d) $W \wedge S$

1. We use the dictionary:

W	it is winter
S	it snows

   What is the best formalization of the sentence:

   It snows because it is winter.

   (a) $W \to S$
   (b) $S \to W$
   (c) $S \leftrightarrow W$
   (d) $W \wedge S$

2. What is the syntax according to the official grammar from the course notes of the formula:

$$\neg a \to b$$

   (a) $\neg(a \to b)$
   (b) $(\neg a \to b)$
   (c) $(\neg(a \to b))$
   (d) $((\neg a) \to b)$

2. What is the syntax according to the official grammar from the course notes of the formula:

$$\neg a \vee \neg b$$

(a) $((\neg a) \vee (\neg b))$
(b) $(\neg a \vee \neg b)$
(c) $\neg(a \vee \neg b)$
(d) $(\neg(a \vee \neg b))$

3. Does the following hold?

$$(a \to b) \to c \equiv a \to (b \to c)$$

(a) Yes, because both formulas are true in the same models.
(b) No, because there is a model in which the left formula is true and the right formula is false (but not the other way around).
(c) No, because there is a model in which the right formula is true and the left formula is false (but not the other way around).
(d) No, because there is a model in which the left formula is true and the right formula is false, and there is a model in which the right formula is true and the left formula is false.

3. Does the following hold?

$$(a \leftrightarrow b) \leftrightarrow c \equiv a \leftrightarrow (b \leftrightarrow c)$$

(a) Yes, because both formulas are true in the same models.
(b) No, because there is a model in which the left formula is true and the right formula is false (but not the other way around).
(c) No, because there is a model in which the right formula is true and the left formula is false (but not the other way around).
(d) No, because there is a model in which the left formula is true and the right formula is false, and there is a model in which the right formula is true and the left formula is false.

4. Consider the sentence:

$$f \equiv g \text{ iff } \models f \leftrightarrow g$$

(where 'iff' is an abbreviation of 'if and only if'). What in this sentence can occur in a formula of propositional logic?

(a) $\equiv$
(b) $\leftrightarrow$ and $\equiv$
(c) $\leftrightarrow$
(d) $\leftrightarrow$ and $\equiv$ and 'iff'

4. Consider the sentence:

$$\text{if } f \models g, \text{ then } \models f \to g$$

What in this sentence can occur in a formula of propositional logic?

(a) $\models$
(b) $\to$ and $\models$
(c) $\to$
(d) $\to$ and $\models$ and 'if ... then'

Propositional logic: open question

5. We use the dictionary:

E	there is an epidemic
H	exams are at home

Give an English sentence without negations that clearly describes the meaning of the propositional formula:

$$\neg(\neg E \vee \neg H)$$

5. We use the dictionary:

E	there is an epidemic
H	exams are at home

Give an English sentence without negations that clearly describes the meaning of the propositional formula:

$$\neg(H \to \neg E)$$

Predicate logic: multiple choice questions

6. We use the dictionary:

M	domain of men
s	Sharon
$N(x)$	x is nice
$L(x,y)$	x loves y

Which of the following formulas corresponds to the sentence:

There is a nice man who loves Sharon.

(a) $\exists x \in M[N(x) \wedge L(x,s)]$
(b) $\exists x \in M[N(x) \to L(x,s)]$
(c) $\forall x \in M[N(x) \wedge L(x,s)]$
(d) $\forall x \in M[N(x) \to L(x,s)]$

6. We use the dictionary:

M	domain of men
s	Sharon
$N(x)$	x is nice
$L(x,y)$	x loves y

Which of the following formulas corresponds to the sentence:

All nice men love Sharon.

(a) $\exists x \in M[N(x) \wedge L(x,s)]$
(b) $\exists x \in M[N(x) \to L(x,s)]$

(c) $\forall x \in M[N(x) \wedge L(x,s)]$

(d) $\forall x \in M[N(x) \rightarrow L(x,s)]$

7. Which of the following formulas does not express that there is exactly one element of D that has property $P(x)$?

   (a) $\exists x \in D\, \forall y \in D\, [P(y) \leftrightarrow y = x]$

   (b) $\exists x \in D\, [P(x) \wedge \forall y \in D\, [y \neq x \rightarrow \neg P(y)]]$

   (c) $\exists x \in D\, P(x) \wedge \forall x_1, x_2 \in D\, [P(x_1) \wedge P(x_2) \rightarrow x_1 = x_2]$

   (d) $\exists x \in D\, \forall y \in D\, [P(y) \rightarrow y = x]$

7. Which of the following formulas does not express that there is at most one element of D that has property $P(x)$?

   (a) $\forall x_1, x_2 \in D\, [P(x_1) \wedge P(x_2) \rightarrow x_1 = x_2]$

   (b) $\forall x_1, x_2 \in D\, [x_1 \neq x_2 \rightarrow \neg P(x_1) \vee \neg P(x_2)]$

   (c) $\neg \exists x_1, x_2 \in D\, [x_1 \neq x_2 \wedge P(x_1) \wedge P(x_2)]$

   (d) $\neg \exists x_1, x_2 \in D\, [P(x_1) \wedge x_1 \neq x_2 \rightarrow \neg P(x_2)]$

8. Consider the following two logical equivalences that express ‘logical laws’:

   - $\neg \forall x \in D\, f \equiv \exists x \in D\, \neg f$
   - $\neg (f \leftrightarrow g) \equiv \neg f \leftrightarrow \neg g$

   Which of these hold?

   (a) both

   (b) only the first

   (c) only the second

   (d) none

8. Consider the following two logical equivalences that express ‘logical laws’:

   - $\neg \exists x \in D\, f \equiv \forall x \in D\, \neg f$
   - $\neg (f \rightarrow g) \equiv \neg g \rightarrow \neg f$

   Which of these hold?

   (a) both

   (b) only the first

   (c) only the second

   (d) none

9. Consider the model (men, women, parent_of), and an interpretation in this model:

M	domain of men
W	domain of women
$P(x,y)$	y is a parent of x

You may assume that both parents of all people in $(M \cup W)$ are also in $(M \cup W)$.

Which of the following formulas is not true in this model?

(a) $\forall x \in (M \cup W)\, \exists y_1 \in M\, \exists y_2 \in W\, [P(y_1, x) \wedge P(y_2, x)]$
(b) $\forall x \in W\, \exists x', x'' \in W\, [P(x, x') \wedge P(x, x'')]$
(c) $\forall x \in W\, \exists x', x'' \in W\, [P(x, x') \wedge P(x', x'')]$
(d) $\forall x \in (M \cup W)\, \exists y \in (M \cup W)\, [P(x, y) \vee P(y, x)]$

9. Consider the model $(\mathsf{men}, \mathsf{women}, \mathsf{parent\_of})$, and an interpretation in this model:

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W	domain of women
$P(x, y)$	y is a parent of x

You may assume that both parents of all people in $(M \cup W)$ are also in $(M \cup W)$.

Which of the following formulas is not true in this model?

(a) $\forall x \in (M \cup W)\, \exists y_1 \in M\, \exists y_2 \in W\, [P(x, y_1) \wedge P(x, y_2)]$
(b) $\forall x \in M\, \exists x', x'' \in M\, [P(x, x') \wedge P(x'', x)]$
(c) $\forall x \in M\, \exists x', x'' \in M\, [P(x, x') \wedge P(x'', x')]$
(d) $\forall x \in (M \cup W)\, \exists y \in (M \cup W)\, [P(x, y) \leftrightarrow P(y, x)]$

Predicate logic: open question

10. Show that:

$$\not\models \forall x, y \in D\, [R(x, y) \vee x = y \vee R(y, x)]$$

Explain your answer.

10. Show that:

$$\not\models [\forall x \in D\, R(x, x)] \vee [\forall x \in D\, \neg R(x, x)]$$

Explain your answer.

Languages: multiple choice questions

11. Let be given a language L. Which of the following languages is not necessarily equal to the others?

(a) LL^*
(b) $LL^* \cup \{\lambda\}$
(c) L^*L^*
(d) L^*

11. Let be given a language L. Which of the following languages is not necessarily equal to the others?

(a) L
(b) $(L^R)^R$

(c) $L \cup (L^R)^R$

(d) $((L^R)^R)^R$

12. Which of the following regular expressions describes a language different from the others?

(a) $(a \cup b)^*$

(b) $(a^*b^*)^*$

(c) $(b^*a^*)^*$

(d) $(a^* \cup b^*)$

12. Which of the following regular expressions describes a language different from the others?

(a) a^*b^*

(b) $(\lambda \cup aa)^*(\lambda \cup bb)^*$

(c) $(a \cup aa)^*(b \cup bb)^*$

(d) $(\lambda \cup a)^*(\lambda \cup b)^*$

13. Consider the following context-free grammar for a fragment of English:

$$\begin{aligned} S &\to N\ \textit{walks} \mid N\ \textit{loves}\ N \mid S\ \textit{and}\ S \\ N &\to \textit{the}\ A\ M \\ A &\to \textit{tall} \mid \textit{small} \mid \lambda \\ M &\to \textit{man} \mid \textit{woman} \end{aligned}$$

Which of the following statements about the language produced by this grammar is not true?

(a) The shortest sentences in the language have three words.

(b) The sentences in this language can be arbitrarily long.

(c) This language contains an infinite sentence.

(d) For each $n \geq 3$ this language contains a sentence of exactly n words.

13. Consider the following context-free grammar for a fragment of a programming language, with alphabet $\Sigma = \{\texttt{:}, \texttt{=}, \texttt{;}, \texttt{0}, \texttt{1}, \texttt{+}, \texttt{*}, \texttt{(}, \texttt{)}, \texttt{x}, \texttt{y}\}$:

$$\begin{aligned} S &\to V \texttt{ := } E \mid S \texttt{ ; } S \\ E &\to \texttt{0} \mid \texttt{1} \mid V \mid E \texttt{ + } E \mid E \texttt{ * } E \mid \texttt{(} E \texttt{)} \\ V &\to \texttt{x} \mid \texttt{y} \end{aligned}$$

Which of the following statements about the language produced by this grammar is not true?

(a) The shortest programs in this language have four symbols.

(b) The programs in this language can be arbitrarily long.

(c) This language contains an infinite program.

(d) This language does not contain a program of five symbols.

14. Which of the following requirements does not necessarily hold for an invariant of a context-free grammar G?

(a) An invariant is a predicate on words from $(\Sigma \cup V)^*$.

(b) An invariant holds for the word $S \in (\Sigma \cup V)^*$.

(c) If an invariant holds for a word, and one symbol in the word is replaced according to a rule from the grammar, the invariant still holds.

(d) The invariant does not hold for the word of which we want to show that it is not in the language $\mathcal{L}(G)$.

14. Which of the following requirements does not necessarily hold for an invariant of a context-free grammar G?

(a) The invariant holds for all words in $\mathcal{L}(G)$.

(b) The invariant holds for all words $S, w_1, w_2, \ldots$ in any production $S \to w_1 \to w_2 \to \ldots$ of the language.

(c) There is a word in $(\Sigma \cup V)^*$ for which the invariant holds.

(d) $\mathcal{L}(G)$ consists of the words in Σ^* that satisfy the invariant.

Languages: open question

15. Give a right linear grammar that produces the same language as the context-free grammar:

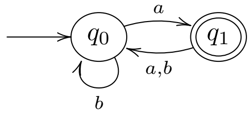
$$\begin{aligned} S &\to AB \\ A &\to aA \mid \lambda \\ B &\to bB \mid \lambda \end{aligned}$$

15. Give a right linear grammar that produces the same language as the context-free grammar:

$$\begin{aligned} S &\to AA \mid \lambda \\ A &\to aaA \mid S \end{aligned}$$

Automata: multiple choice questions

16. Consider the following DFA:



How many words of four letters does this automaton accept?

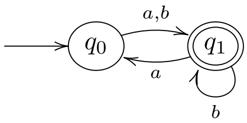
(a) three

(b) four

(c) less than three

(d) more than four

16. Consider the following DFA:



How many words of four letters does this automaton accept?

(a) three
(b) four
(c) less than three
(d) more than four

17. What is the minimum number of states for a DFA that accepts the language

$$\mathcal{L}\big((a \cup b)^* a (a \cup b)^*\big)$$

(a) two
(b) three
(c) less than two
(d) more than three

17. What is the minimum number of states for a DFA that accepts the language

$$\mathcal{L}\big(\lambda \cup aba^*\big)$$

(a) two
(b) three
(c) less than two
(d) more than three

18. Consider the NFA:

$$M := \langle \Sigma, Q, q_0, F, \delta \rangle$$

with:

$$\Sigma = \{a, b\}$$
$$Q = \{q_0, q_1\}$$
$$F = \{q_0\}$$

$$\delta(q_0, a) = \{q_0, q_1\}$$
$$\delta(q_0, b) = \varnothing$$
$$\delta(q_0, \lambda) = \varnothing$$
$$\delta(q_1, a) = \varnothing$$
$$\delta(q_1, b) = \{q_1\}$$
$$\delta(q_1, \lambda) = \{q_0\}$$

What is true?

(a) $aab \in L(M)$ and $baa \in L(M)$
(b) $aab \in L(M)$ and $baa \notin L(M)$
(c) $aab \notin L(M)$ and $baa \in L(M)$
(d) $aab \notin L(M)$ and $baa \notin L(M)$

18. Consider the NFA:
$$M := \langle \Sigma, Q, q_0, F, \delta \rangle$$
with:
$$\begin{aligned} \Sigma &= \{a, b\} \\ Q &= \{q_0, q_1\} \\ F &= \{q_0\} \end{aligned}$$
$$\begin{aligned} \delta(q_0, a) &= \{q_0, q_1\} \\ \delta(q_0, b) &= \varnothing \\ \delta(q_0, \lambda) &= \varnothing \\ \delta(q_1, a) &= \varnothing \\ \delta(q_1, b) &= \{q_1\} \\ \delta(q_1, \lambda) &= \{q_0\} \end{aligned}$$
What is true?
(a) $aba \in L(M)$ and $bab \in L(M)$
(b) $aba \in L(M)$ and $bab \notin L(M)$
(c) $aba \notin L(M)$ and $bab \in L(M)$
(d) $aba \notin L(M)$ and $bab \notin L(M)$

19. There exist DFAs with 2020 states that accept the word $a^{2021}b^{2021}$. Does such a DFA always accept a word $a^n b^{2021}$ as well, for some $n > 2021$?
(a) Yes, because while processing the a's in $a^{2021}b^{2021}$, there has to be a state that occurs twice, which means there is a loop while processing the a's.
(b) No, because it only accepts words of the form $a^n b^n$.
(c) No, because the language $\{a^n b^n \mid n \in \mathbb{N}\}$ is not regular, so it is not accepted by a DFA.
(d) You cannot know this, this is the case for some of these automata, but not for all.

19. There exist DFAs with 2020 states that accept the word $a^{2021}b^{2021}$. Does such a DFA always accept a word $a^{2021}b^n$ as well, for some $n > 2021$?
(a) Yes, because while processing the b's in $a^{2021}b^{2021}$, there has to be a state that occurs twice, which means there is a loop while processing the b's.
(b) No, because it only accepts words of the form $a^n b^n$.
(c) No, because the language $\{a^n b^n \mid n \in \mathbb{N}\}$ is not regular, so it is not accepted by a DFA.
(d) You cannot know this, this is the case for some of these automata, but not for all.

Automata: open question

20. Give a regular language with alphabet $\Sigma = \{a, b\}$ that does not contain the empty word, and for which there does not exist a DFA with at most two states.

20. Give a regular language with alphabet $\Sigma = \{a, b\}$ that does contain the empty word, and for which there does not exist a DFA with at most two states.

Discrete mathematics: multiple choice questions

21. For which $n \geq 1$ is K_n a tree?

    (a) For $n = 1$.
    (b) For $n = 2$.
    (c) For $n = 1$ and $n = 2$.
    (d) This is never a tree.

21. For which $n, m \geq 1$ is $K_{n,m}$ a tree?

    (a) For $n = 1$ and any m, or $m = 1$ and any n.
    (b) For $n = 1$ and $m = 1$.
    (c) For $n = m$.
    (d) This is never a tree.

22. We define:

$$\begin{aligned} a_1 &= 1 \\ a_{n+1} &= a_n + n + 1 \qquad \text{for } n \geq 1 \end{aligned}$$

    What is a_4?

    (a) 8
    (b) 11
    (c) 13
    (d) None of the above.

22. We define:

$$\begin{aligned} a_1 &= 0 \\ a_{n+1} &= a_n + n - 1 \qquad \text{for } n \geq 1 \end{aligned}$$

    What is a_4?

    (a) 3
    (b) 5
    (c) 6
    (d) None of the above.

23. We have a proof by induction that shows that a predicate $P(n)$ holds for all n starting at zero. This proof follows the standard scheme, in which the base case just is about $P(0)$. What from this proof is used to establish that $P(3)$ holds?

   (a) The proof of the base case, and the induction steps for $k = 0$, $k = 1$ and $k = 2$.

   (b) The proof of the base case, and the induction steps for $k = 0$, $k = 1$, $k = 2$ and $k = 3$.

   (c) The base case and the induction step for $k = 2$.

   (d) The base case and the induction step for $k = 3$.

23. We want to prove that a predicate $P(n)$ holds for all $n \geq 0$, but we only manage to prove the induction step for $k \geq 2$. What can we do?

   (a) We cannot use induction to prove this, because in an induction proof the induction step needs to start at the same index as the statement that we want to prove.

   (b) We can still use induction to prove this, if next to the base case for $P(0)$ we manage to prove extra base cases for $P(1)$ and $P(2)$.

   (c) We can still use induction to prove this, if next to the base case for $P(0)$ we manage to prove an extra base case for $P(1)$.

   (d) We can still use induction to prove this, we just use the base case $P(2)$ instead of $P(0)$.

24. We want to count the number of ways that one can divide nine distinguishable objects into four non-distinguishable (possibly empty) groups. What can we best use for this?

   (a) Binomial coefficients.

   (b) Stirling numbers of the first kind.

   (c) Stirling numbers of the second kind.

   (d) Bell numbers.

24. We want to count the number of ways that one can select a non-empty selection of at most four objects out of nine distinguishable objects. What can we best use for this?

   (a) Binomial coefficients.

   (b) Stirling numbers of the first kind.

   (c) Stirling numbers of the second kind.

   (d) Bell numbers.

Discrete mathematics: open question

25. Give a planar connected graph that has an Eulerian circuit, but not a Hamiltonian circuit.

25. Give a planar connected graph that has an Eulerian path, but not a Hamiltonian path.

Modal logic: multiple choice questions

26. What is the appropriate logic to formalize a statement of the form:

    This is allowed, but not required.

    (a) Epistemic logic.
    (b) Doxastic logic.
    (c) Deontic logic.
    (d) Alethic logic (the logic of necessity and possibility).

26. What is the appropriate logic to formalize a statement of the form:

    This might be the case, but it also might not be the case.

    (a) Epistemic logic.
    (b) Doxastic logic.
    (c) Deontic logic.
    (d) Alethic logic (the logic of necessity and possibility).

27. In the logic T, the axiom scheme $\Box f \to f$ holds for all formulas f. Is this logic appropriate for doxastic logic?

    (a) No, because it is possible to believe false things.
    (b) No, because it is possible not to do things that are obligatory.
    (c) Yes, because it is not possible to believe contradictory things.
    (d) Yes, because it is not possible to require something that is forbidden.

27. In the logic T, the axiom scheme $\Box f \to f$ holds for all formulas f. Is this logic appropriate for deontic logic?

    (a) No, because it is possible to believe false things.
    (b) No, because it is possible not to do things that are obligatory.
    (c) Yes, because it is not possible to believe contradictory things.
    (d) Yes, because it is not possible to require something that is forbidden.

28. If in a Kripke model we have $V(x_i) = \varnothing$, then what is necessarily the case?

    (a) $x_i \Vdash \Box\neg\Box f$ for all formulas f.
    (b) $x_i \Vdash \Diamond\neg\Box f$ for all formulas f.
    (c) $x_i \Vdash \neg\Box\Diamond f$ for all formulas f.
    (d) $x_i \Vdash \Diamond\Diamond\neg f$ for all formulas f.

28. If in a Kripke model we have $V(x_i) = \varnothing$, then what is necessarily the case?

    (a) $x_i \Vdash \neg\Box\Box f$ for all formulas f.
    (b) $x_i \Vdash \Diamond\Box\neg f$ for all formulas f.
    (c) $x_i \Vdash \Box\Diamond\neg f$ for all formulas f.

(d) $x_i \Vdash \Diamond\neg\Diamond f$ for all formulas f.

29. Which of the following is true in every LTL model?

    (a) $a\,\mathcal{U}\,b \to a$
    (b) $a\,\mathcal{U}\,b \to \mathcal{F}b$
    (c) $a \to a\,\mathcal{U}\,b$
    (d) $\mathcal{F}b \to a\,\mathcal{U}\,b$

29. Which of the following is true in every LTL model?

    (a) $a\,\mathcal{W}\,b \to \mathcal{F}a$
    (b) $a\,\mathcal{W}\,b \to \mathcal{G}a$
    (c) $\mathcal{F}a \to a\,\mathcal{W}\,b$
    (d) $\mathcal{G}a \to a\,\mathcal{W}\,b$

Modal logic: open question

30. Give a serial Kripke model $\mathcal{M}$ with:

$$\mathcal{M} \not\models a \to \Diamond a$$

Explain your answer, using both the symbols $\models$ and $\Vdash$.

30. Give a non-reflexive Kripke model $\mathcal{M}$ with:

$$\mathcal{M} \models a \to \Diamond a$$

Explain your answer, using both the symbols $\models$ and $\Vdash$.