Formal Reasoning 2020
Solutions Test Block 1: Propositional and Predicate Logic
(21/09/20)

Propositional logic

1. We use the dictionary

C	there is a pandemic
M	this test has multiple choice questions

Which of the following formulas of propositional logic is the best formalization of the following English sentence:

> *This test has multiple choice questions, both when there is a pandemic and when there is not a pandemic.*

(a) $(C \to M) \land (\neg C \to M)$

This means something like:

> *If there is a pandemic then this test has multiple choice questions, and if there is no pandemic then this test has multiple choice questions.*

This is the best formalization.

(b) $(C \to M) \lor (\neg C \to M)$

This means something like:

> *If there is a pandemic then this test has multiple choice questions, or if there is no pandemic then this test has multiple choice questions.*

(c) $(C \land \neg C) \leftrightarrow M$

This means something like:

> *There is a pandemic and there is not a pandemic, if and only if this test has multiple choice questions.*

(d) $(C \land \neg C) \to M$

This means something like:

> *If there is a pandemic and there is not a pandemic, then this test has multiple choice questions.*

1. We use the dictionary

C	there is a pandemic
M	this test has multiple choice questions

Which of the following formulas of propositional logic is the best formalization of the following English sentence:

> *This test has multiple choice questions, because there is a pandemic.*

(a) $M \to C$

This means something like:

If this test has multiple choice questions, then there is a pandemic.

(b) $C \to M$

This means something like:

If there is a pandemic, then this test has multiple choice questions.

(c) $C \leftrightarrow M$

This means something like:

There is a pandemic if and only if this test has multiple choice questions.

(d) $C \wedge M$

This means something like:

There is a pandemic and this test has multiple choice questions.

This is the best formalization. Note that 'because' typically implies a conjunction.

2. The truth table of the formula of propositional logic

$$a \wedge \neg b \leftrightarrow c$$

consists of eight rows. How many of those rows have a 0?

(a) 1

(b) 2

(c) 4

This is the correct answer, as can be derived from this truth table:

a	b	c	$\neg b$	$a \wedge \neg b$	$a \wedge \neg b \leftrightarrow c$
0	0	0	1	0	1
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	0	0	0
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	0

(d) None of the above.

2. The truth table of the formula of propositional logic

$$a \wedge \neg b \vee c$$

consists of eight rows. How many of those rows have a 0?

(a) 1

(b) 2

(c) 4

(d) None of the above.

This is the correct answer, as can be derived from this truth table:

a	b	c	$\neg b$	$a \wedge \neg b$	$a \wedge \neg b \to c$
0	0	0	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	0	1

3. A model of propositional logic corresponds in the truth table to . . .

(a) One of the rows.

A model is a valuation that assigns truth values to atomic propositions. So each model is a row in the truth table.

(b) One of the columns.

(c) The final column for the whole formula.

(d) The whole table.

3. How many rows are there in the truth table of a propositional formula that contains only a single atomic proposition a?

(a) 1

(b) 2

Each row in the truth table corresponds to a model. A model is a valuation that assigns truth values to atomic propositions. If there is only one atomic proposition, then we only have to choose once between 0 and 1, so that gives two rows. In particular, it does not depend on the formula. The formula determines the number of columns.

(c) 4

(d) It depends on the formula.

4. Does the following hold?

$$\neg(a \to b) \equiv \neg a \to \neg b$$

(a) Yes, this is one of the distributive laws.

It doesn't hold. . .

(b) Yes, this is the law of contraposition.

It doesn't hold. . .

(c) No, you can see this by comparing two columns in an appropriate truth table.

Correct: check whether the columns for $\neg(a \to b)$ and $\neg a \to \neg b$ are exactly the same.

a	b	$a \to b$	$\neg(a \to b)$	$\neg a$	$\neg b$	$\neg a \to \neg b$
0	0	1	0	1	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	1	1	0	0	0	1

The columns are not exactly the same. So the formulas are not logically equivalent.

(d) No, because there is a model in which this logical equivalence is not true.

The property of being 'logically equivalent' does not depend on a particular model. It makes no sense to state that this *logical equivalence* holds in a model or not. Note the difference with stating that two formulas have the same truth value in a certain model, which does make sense.

4. Does the following hold?

$$\neg(a \to b) \models \neg a \to \neg b$$

(a) Yes, this follows from one of the distributive laws.

It holds, but we didn't define any distributive laws.

(b) Yes, you can see this by comparing two columns in an appropriate truth table.

Correct: check whether the column for $\neg a \to \neg b$ has a 1 on all rows where the column for $\neg(a \to b)$ has a 1.

a	b	$a \to b$	$\neg(a \to b)$	$\neg a$	$\neg b$	$\neg a \to \neg b$
0	0	1	0	1	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	1
1	1	1	0	0	0	1

The formula $\neg(a \to b)$ only has a 1 in the third row. And the column for $\neg a \to \neg b$ also has a 1 in that row. So the second formula is a logical consequence of the first.

(c) No, because the law of contraposition gives that $a \to b \models \neg b \to \neg a$ instead of $a \to b \models \neg a \to \neg b$.

It holds...

(d) No, because there is a model in which this logical consequence is not true.

It holds...

5. Which of the following holds?

(a) If $\vDash f \vee g$ then $\vDash f$ or $\vDash g$ (or both).

This doesn't hold. Take $f = a$ and $g = \neg a$. Then clearly $\vDash f \vee g$, but also clearly $\nvDash f$ and $\nvDash g$.

(b) If $f \vee g \equiv h$ then $f \equiv h$ or $g \equiv h$ (or both).

This doesn't hold. Take $f = a$, $g = \neg a$, and $h = a \vee nega$. Then clearly $f \vee g \equiv h$, but also clearly $f \not\equiv h$ and $g \not\equiv h$.

(c) If $\vDash f$ or $\vDash g$ (or both) then $\vDash f \vee g$.

This holds. If $\vDash f$ holds, then f has a 1 on every row in its truth table. But then automatically there is also a 1 on every row in the truth table of $f \vee g$. So $\vDash f \vee g$ holds. Likewise, if $\vDash g$ holds, then g has a 1 on every row in its truth table. But then automatically there is also a 1 on every row in the truth table of $f \vee g$. So $\vDash f \vee g$ holds. These two cases already include the 'both case' as well, so in all cases we see that the statement holds.

(d) If $f \equiv h$ or $g \equiv h$ (or both) then $f \vee g \equiv h$.

This doesn't hold. Let $f = a$, $g = \neg a$, and $h = a$. Then clearly $f \equiv h$. But $f \vee g \not\equiv h$, because h has a 0 and a 1 in its column, whereas $f \vee g$ only has 1s.

5. Which of the following does *not* hold?

(a) If $\vDash f \wedge g$ then $\vDash f$ and $\vDash g$.

This holds. From $\vDash f \wedge g$ it follows that every row for $f \wedge g$ has a 1. But that can only be if every row for both f and g has a 1. So $\vDash f$ and $\vDash g$ hold.

(b) If $f \wedge g \equiv h$ then $f \equiv h$ and $g \equiv h$.

This doesn't hold. Let $f = a$, $g = \neg a$ and $h = a \wedge \neg a$. Then clearly $f \wedge g \equiv h$, but $f \not\equiv h$ because f has a 1 and h does not, and $g \not\equiv h$ because g has a 1 and h does not.

(c) If $\vDash f$ and $\vDash g$ then $\vDash f \wedge g$.

This holds. From $\vDash f$ it follows that f has a 1 on every row. From $\vDash g$ it follows that g has a 1 on every row. But then it follows that $f \wedge g$ has a 1 on every row. And this implies $\vDash f \wedge g$.

(d) If $f \equiv h$ and $g \equiv h$ then $f \wedge g \equiv h$.

This holds. On rows where h has a 0, we know from $f \equiv h$ and $g \equiv h$ that both f and g have a 0 as well, and hence $f \wedge g$ has a 0. On rows where h has a 1, we know from $f \equiv h$ and $g \equiv h$ that both f and g have a 1 as well, and hence $f \wedge g$ has a 1. So in all cases we see that the truth value of h is equal to the truth value of $f \wedge g$ on every row. So $f \wedge g \equiv h$.

Predicate logic

6. What is the syntax of the following formula according to the official grammar from the course notes?

$$(\exists x_1, x_2 \in D\,(x_1 \neq x_2))$$

(a) $(\exists x_1 \in D\,(\exists x_2 \in D\,(\neg(x_1 = x_2))))$
This is not correct because of the parentheses around the negation.

(b) $(\exists x_1 \in D\,(\exists x_2 \in D\,\neg(x_1 = x_2)))$
This is correct: parentheses around the equality, no parentheses around the negation, parentheses around the existential quantification, and parentheses around the universal quantification.

(c) $(\exists x_1 \in D\,[\exists x_2 \in D\,[\neg(x_1 = x_2)]])$
This is not correct because of the usage of square brackets.

(d) $\exists x_1 \in D\,[\exists x_2 \in D\,[\neg(x_1 = x_2)]]$
This is not correct because of the usage of square brackets.

6. What is the syntax of the following formula according to the official grammar from the course notes?

$$\forall x \in D\,[\exists y \in D\,R(x,y) \wedge P(x)]$$

(a) $\forall x \in D\,[\exists y \in D\,[(R(x,y) \wedge P(x))]]$
This is not correct because of the usage of square brackets.

(b) $\forall x \in D\,[(\exists y \in D\,[R(x,y)] \wedge P(x))]$
This is not correct because of the usage of square brackets.

(c) $(\forall x \in D\,(\exists y \in D\,(R(x,y) \wedge P(x))))$
This is not correct because of the strong binding of the existential quantification.

(d) $(\forall x \in D\,((\exists y \in D\,R(x,y)) \wedge P(x)))$
This is correct because of the strong binding of the existential quantification. In addition it has parentheses around the existential quantification, around the conjunction, and around the universal quantification.

7. We use the dictionary

P	domain of people
V	domain of vehicles
$F(x)$	x wears a face mask
$T(x)$	x is public transport vehicle
$I(x,y)$	x is inside y

Which of the following formulas of predicate logic is the best formalization of the following English sentence:

In public transport, people wear a face mask.

(a) $\forall x \in P\,\forall y \in V\,(T(y) \wedge I(x,y) \to F(x))$
This is a correct formalization. Literally translated we get:

For every person and vehicle it holds that if this vehicle is a public transport vehicle and this person is inside it, then this person wears a face mask.

And this is basically the same as the original sentence.

(b) $\forall x \in P\, \forall y \in V\, (T(y) \wedge I(x,y) \wedge F(x))$

This means something like:

Every vehicle is a public transport, every person wears a face mask, and every person is inside every vehicle.

(c) $\forall x \in P\, \exists y \in V\, (T(y) \wedge I(x,y) \wedge F(x))$

This means something like:

For every person there exists a public transport vehicle for which it holds that this person is inside it, and this person wears a face mask.

(d) $\forall x \in P\, ((\forall y \in V\, (T(y) \wedge I(x,y))) \rightarrow F(x))$

This means something like:

For every person it holds that if every vehicle is a public transport vehicle and this person is inside it, then this person wears a face mask.

7. We use the dictionary

P	domain of people
V	domain of vehicles
$F(x)$	x wears a face mask
$T(x)$	x is public transport vehicle
$I(x,y)$	x is inside y

Which of the following formulas of predicate logic is the best formalization of the following English sentence:

If a person does not wear a face mask, he or she is not traveling by public transport.

(a) $\forall x \in P\, (\neg F(x) \rightarrow \neg \forall y \in V (T(y) \wedge I(x,y)))$

This means something like:

For every person it holds that if this person is not wearing a face mask, it is not the case that all vehicles are public transport vehicles and this person is inside them.

(b) $\forall x \in P\, (\neg F(x) \rightarrow \neg \exists y \in V (T(y) \wedge I(x,y)))$

This means something like:

For every person it holds that if this person is not wearing a face mask, it is not the case that there is a vehicle which is a public transport vehicle and this person is inside it.

This is basically the same as the original sentence.

(c) $\exists x \in P\,(\neg F(x) \to \neg\forall y \in V(T(y) \wedge I(x, y)))$
This means something like:

There exists a person for which it holds that if this person is not wearing a face mask, it is not the case that all vehicles are public transport vehicles and this person is inside them.

(d) $(\exists x \in P\,\neg F(x)) \to \neg\exists y \in V(T(y) \wedge I(x, y))$
This means something like:

If there is a person who is not wearing a face mask, then ...

Because the last x is outside the scope of the $\exists x \in P$, there is no way to translate this into a sentence, because we don't know who this x is.

8. How does one express in predicate logic with equality, that a domain D has exactly two elements that satisfy a predicate P?

(a) $\exists x_1, x_2 \in D\,[x_1 \neq x_2 \wedge P(x_1) \wedge P(x_2)]$
This is incorrect because there can be more than two elements satisfying predicate P.

(b) $\exists x_1, x_2 \in D\,[x_1 \neq x_2 \wedge \forall y \in D\,[P(y) \leftrightarrow y = x_1 \vee y = x_2]]$
This is correct.

(c) $\exists x_1, x_2 \in D\,[x_1 \neq x_2 \wedge \forall y \in D\,[P(y) \to y = x_1 \vee y = x_2]]$
This is incorrect because it doesn't specify that the elements x_1 and x_2 actually satisfy predicate P.

(d) $\exists x_1, x_2 \in D\,[x_1 \neq x_2] \wedge \forall y \in D\,[P(y) \to y = x_1 \vee y = x_2]$
This is incorrect because of the wrong scope for x_1 and x_2. The x_1 and x_2 in the universal quantification are not bound by the first existential quantification.

8. How does one express in predicate logic with equality, that a domain D does not have exactly one element that satisfies a predicate P (i.e., there are no elements for which P holds, or two or more)?

(a) $\forall x \in D[P(x) \to \exists y \in D[P(y)]]$
This is incorrect because it allows for exactly one element satisfying predicate P, by taking $y = x$.

(b) $\forall x, y \in D[P(x) \wedge P(y) \to x \neq y]$
This is incorrect because it implies that there are no elements satisfying predicate P.

(c) $(\neg\exists x \in D[P(x)]) \vee (\exists x_1, x_2 \in D[P(x_1) \wedge P(x_2)])$
This is incorrect because it allows for exactly one element satisfying predicate P, by taking $x_1 = x_2$.

(d) $\neg\exists x \in D[P(x) \wedge \forall y \in D[P(y) \to x = y]]$
This is correct because it is simply the negation of the default pattern for stating that there is exactly one element satisfying predicate P.

9. We define a model

$$M = (\mathbb{N}, <)$$

and an interpretation I that maps N to $\mathbb{N}$ and $L(x, y)$ to $x < y$. Which of the following statements hold?

(a) $(M, I) \vDash \exists x \in N\, \forall y \in N\, L(x, y)$

This is incorrect because it would imply that there is a natural number which is smaller than all natural numbers, but there is no natural number that is smaller than 0.

(b) $(M, I) \vDash \exists x \in N\, \forall y \in N\, L(y, x)$

This is incorrect because it would imply that there is a natural number for which it holds that all natural numbers are smaller than this one, but for every candidate x we can take $y = x + 1$ and then the formula doesn't hold.

(c) $(M, I) \vDash \forall x \in N\, \exists y \in N\, L(x, y)$

This is correct because it means that for all natural numbers x there is a natural number y such that $x < y$. Take $y = x + 1$ and it works.

(d) $(M, I) \vDash \forall x \in N\, \exists y \in N\, L(y, x)$

This is incorrect because it implies that for all natural numbers x there is a natural number y such that $y < x$, but there is no such y if we take $x = 0$.

9. We define a model

$$M = (\mathbb{N}, \leq)$$

and an interpretation I that maps N to $\mathbb{N}$ and $L(x, y)$ to $x \leq y$. Which of the following statements does *not* hold?

(a) $(M, I) \vDash \exists x \in N\, \forall y \in N\, L(x, y)$

This is correct because it implies that there exists a natural number x for which it holds that it is less than or equal to all natural numbers. This is the case for $x = 0$.

(b) $(M, I) \vDash \exists x \in N\, \forall y \in N\, L(y, x)$

This is incorrect because it implies that there is a natural number for which it holds that all natural numbers are less than or equal to this number. However, for every candidate x that we have, we can take $y = x + 1$ and then $y \leq x$ doesn't hold.

(c) $(M, I) \vDash \forall x \in N\, \exists y \in N\, L(x, y)$

This is correct because it implies that for all natural numbers x there is a natural number y such that $y \leq x$. Take $y = 0$ and it works.

(d) $(M, I) \vDash \forall x \in N\, \exists y \in N\, L(y, x)$

This is correct because it implies that for all natural numbers x we can find a natural number y such that $x \leq y$. Take $y = x + 1$ and it works.

10. Which of the following statements is correct?

(a) $\neg\forall x \in D\,(P(x) \to Q(x)) \equiv \forall x \in D\,(\neg P(x) \to \neg Q(x))$
This is incorrect because a negation of a 'for all' always leads to an 'exists'.

(b) $\neg\forall x \in D\,(P(x) \to Q(x)) \equiv \forall x \in D\,(\neg(P(x) \to Q(x)))$
This is incorrect because a negation of a 'for all' always leads to an 'exists'.

(c) $\neg\forall x \in D\,(P(x) \to Q(x)) \equiv \exists x \in D\,(P(x) \wedge \neg Q(x))$
This is correct because

$$\begin{aligned}
\neg\forall x \in D\,(P(x) \to Q(x)) &\equiv \exists x \in D\,\neg(P(x) \to Q(x)) \\
&\equiv \exists x \in D\,\neg(\neg P(x) \vee Q(x)) \\
&\equiv \exists x \in D\,(\neg\neg P(x) \wedge \neg Q(x)) \\
&\equiv \exists x \in D\,(P(x) \wedge \neg Q(x))
\end{aligned}$$

(d) $\neg\forall x \in D\,(P(x) \to Q(x)) \equiv \exists x \in D\,(\neg P(x) \wedge \neg Q(x))$
This is incorrect because the $\neg P(x)$ should have been simply $P(x)$ as can be seen above.

10. Which of the following statements is correct?

(a) $\neg\exists x \in D\,(P(x) \wedge Q(x)) \equiv \exists x \in D\,(\neg P(x) \wedge \neg Q(x))$
This is incorrect because a negation of an 'exists' always leads to a 'for all'.

(b) $\neg\exists x \in D\,(P(x) \wedge Q(x)) \equiv \exists x \in D\,(\neg(P(x) \wedge Q(x)))$
This is incorrect because a negation of an 'exists' always leads to a 'for all'.

(c) $\neg\exists x \in D\,(P(x) \wedge Q(x)) \equiv \forall x \in D\,(P(x) \to \neg Q(x))$
This is incorrect because

$$\begin{aligned}
\neg\exists x \in D\,(P(x) \wedge Q(x)) &\equiv \forall x \in D\,\neg(P(x) \wedge Q(x)) \\
&\equiv \forall x \in D\,(\neg P(x) \vee \neg Q(x)) \\
&\equiv \forall x \in D\,(P(x) \to \neg Q(x))
\end{aligned}$$

(d) $\neg\exists x \in D\,(P(x) \wedge Q(x)) \equiv \forall x \in D\,(\neg P(x) \to \neg Q(x))$
This is incorrect because the $\neg P(x)$ should have been simply $P(x)$ as can be seen above.

Open question

11. Give an interpretation in a model that makes the following formula true:

$$(\exists x \in D\ \ R(x,x)) \wedge \forall y_1, y_2 \in D\,\exists z \in D\,(R(y_1,z) \wedge R(z,y_2))$$

Use an interpretation for D that is a set of numbers.

You do *not* need to explain why your model makes this formula true.

We can take as a model:

Domain(s):	Natural numbers ($\mathbb{N}$) (But $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ will also work with the given interpretation.)
Predicate(s):	None
Relation(s):	equality, greater than, less than, greater than or equal, less than or equal

We take as interpretation:

D	$\mathbb{N}$
$R(x, y)$	$x \cdot y = 0$

The formula now states that

- There is a natural number whose square is 0: this is the case if $x = 0$.
- For any two numbers y_1 and y_2 it holds that there exists a number z such that $y_1 \cdot z = 0$ and $z \cdot y_2 = 0$, which holds because we can take $z = 0$.

Other interpretations that work (we keep the models implicit)

- $D \mapsto \mathbb{N}$ and $R(x, y) \mapsto$ true.
- $D \mapsto \mathbb{N}$ and $R(x, y) \mapsto (x \leq y$ or $x \geq y)$.
- $D \mapsto \mathbb{N}$ and $R(x, y) \mapsto x \geq 0$. Note that this one won't work if you take for instance $\mathbb{Z}$ as domain. And it feels a bit like cheating if you take a binary relation that does not really take its second argument into account.
- $D \mapsto \{1\}$ and $R(x, y) \mapsto x = y$.

11. Give an interpretation in a model that makes the following formula true:

$$(\exists x \in D \, \neg R(x, x)) \wedge \forall y \in D \, (\exists z_1 \in D \, R(y, z_1) \wedge \exists z_2 \in D \, R(z_2, y))$$

Use an interpretation for D that is a set of numbers.

You do *not* need to explain why your model makes this formula true.

We can take as a model:

Domain(s):	Integers ($\mathbb{Z}$) (Note that $\mathbb{Q}$ and $\mathbb{R}$ will also work with the given interpretation, but $\mathbb{N}$ won't.)
Predicate(s)	None
Relation(s):	equality, greater than, less than, greater than or equal, less than or equal

We take as interpretation:

D	$\mathbb{Z}$
$R(x, y)$	$x < y$

The formula now states that

- There is no integer that is smaller than itself, which is obviously true.
- For any integer y there exist two integers z_1 and z_2 such that $y < z_1$ and $z_2 < y$, which holds because we can take $z_1 = y+1$ and $z_2 = y-1$.

Other interpretations that work (we keep the models implicit)

- $D \mapsto \mathbb{Z}$ and $R(x, y) \mapsto x > y$.
- $D \mapsto \{1, 2\}$ and $R(x, y) \mapsto x \neq y$.