Formal Reasoning 2020
Solutions Test Block 3: Discrete Mathematics and Modal Logic

(9/12/20)

Multiple choice questions

1. A non-empty tree with n vertices has always exactly $n-1$ edges. A *forest* is a graph that has no cycles. Therefore a forest is what you would expect: a graph in which each component is a tree.

   Which of the following formulas is the generalization of the edge count formula to forests?

(a) is correct

   (a) A forest with n vertices and c components has $n-c$ edges.

   (b) A forest with n vertices and c components has $(n-1)+c$ edges.

   (c) A forest with n vertices and c components has $n+c-2$ edges.

   (d) A forest with n vertices has $n-1$ edges; the number of components does not matter.

   Answer (a) is correct. Note that a forest is a graph where each component is a tree. So if a forest has c components, we know that for each component there exists a natural number n_i equal to its number of vertices. So $n = n_1 + n_2 + \cdots + n_c$ and component i has by the given formula $n_i - 1$ edges. So the total number of edges in the graph is

$$\sum_{i=1}^{c}(n_i - 1) = \sum_{i=1}^{c} n_i - \sum_{i=1}^{c} 1 = \sum_{i=1}^{c} n_i - c = n - c$$

   It is clear that none of the other options can be correct at the same time.

1. Which of the following statements correctly characterizes the number of edges in a non-empty connected graph, in the sense that the bounds hold and are optimal?

(a) is correct

   (a) A non-empty connected graph with n vertices has at least $n-1$ edges and at most $\frac{1}{2}n(n-1)$ edges.

   (b) A non-empty connected graph with n vertices has at least n edges and at most $\frac{1}{2}n(n-1)$ edges.

   (c) A non-empty connected graph with n vertices has at least $n-1$ edges and at most $\frac{1}{2}n(n+1)$ edges.

   (d) A non-empty connected graph with n vertices has at least n edges and at most $\frac{1}{2}n(n+1)$ edges.

   Answer (a) is correct. The upperbound is needed when we have K_n and in exercise 5.C we have shown that this bound is $\frac{1}{2}n(n-1)$, because for each of the n vertices we have $n-1$ edges to all $n-1$ other vertices, but this method counts every edge twice, so we multiply with $\frac{1}{2}$. The lowerbound should hence be n or $n-1$. But it cannot be n, because the connected graph with one vertex has zero edges. So it must be $n-1$.

   It is clear that none of the other options can be correct at the same time.

2. Two graphs G_1 and G_2 are isomorphic and G_1 is planar. Is G_2 then also always planar?

(a) is correct

(a) Yes, it does not matter for planarity what the labels of the vertices are.
(b) Yes, because G_1 and G_2 have the same chromatic number.
(c) No, if G_1 is drawn without crossing edges, then you don't know whether G_2 has crossing edges or not.
(d) No, it depends on the graphs G_1 and G_2 whether this is the case.

Answer (a) is correct. Being planar depends on the structure of the graph. And if G_1 and G_2 are isomorphic, it means that the structure is exactly the same, but the labels of the vertices may be different.

The second answer makes no sense because the chromatic number has no direct relation with being isomorphic. If two graphs are isomorphic, then they have the same chromatic number, but it doesn't hold in the other direction. Graphs may have the same chromatic number, but don't have to be isomorphic.

The third and the fourth answer make no sense because the answer is 'yes'.

2. How many isomorphisms are there from $K_{2,3}$ to itself?

(c) is correct

(a) Only one.
(b) Six.
(c) Twelve.
(d) None of the above.

Answer (c) is correct. The graph $K_{2,3}$ is a complete bipartite graph with, say, two 'red' vertices and three 'blue' vertices. So the red vertices have degree three and the blue vertices have degree two. If an isomorphism would map a red vertex onto a blue vertex its degree would change, which is not allowed. So an isomorphism can only permute the two red vertices between themselves and the three blue vertices between themselves. For the first permutation there are 2! possibilities and for the second permutation 3!. So in total there are $2! \cdot 3! = 2 \cdot 6 = 12$ possibilities for an isomorphism from $K_{2,3}$ to itself.

It is clear that none of the other options can be correct at the same time.

3. We want to give a recursive definition of $n! = 1 \cdot 2 \cdot \ldots \cdot n$, using equations:

$$\begin{aligned} 0! &= \ldots \\ (n+1)! &= n! \cdot (n+1) \qquad\qquad \text{for } n \geq 0 \end{aligned}$$

For which value for 0! will this recursive definition work?

(b) is correct

(a) $0! = 0$
(b) $0! = 1$
(c) It does not matter what value for 0! one takes, the values of $n!$ for $n \geq 0$ will not change because of that.

(d) This is not possible, because if one uses the recursive equation backwards to calculate 0! from 1!, one is dividing by zero.

Answer (b) is correct. From the definition $n! = 1 \cdot 2 \cdot \ldots \cdot n$, it follows that $1! = 1$. So we get

$$1! = (0+1)! = 0! \cdot (0+1) = 0! \cdot 1 = 1$$

So it must be that $0! = 1$.

The first answer makes no sense because this would imply that

$$1! = (0+1)! = 0! \cdot (0+1) = 0! \cdot 1 = 0 \cdot 1 = 0$$

whereas we know it should be 1.

The third answer makes no sense because the answer is irrelevant with respect to the question.

The fourth answer makes no sense because there is no division by zero, but a division by one.

3. We want to give a recursive definition of a^n, using equations:

$$\begin{aligned} a^0 &= \ldots \\ a^{n+1} &= a^n \cdot a & \text{for } n \geq 0 \end{aligned}$$

For which value for a^0 will this recursive definition work?

(a) $a^0 = 0$

(b) is correct

(b) $a^0 = 1$

(c) It does not matter what value for a^0 one takes, the values of a^n for $n \geq 0$ will not change because of that.

(d) This is not possible, because a^0 is the multiplication of zero a's, and you cannot multiply anything when there are no a's to be multiplied.

Answer (b) is correct. We know that $a^1 = a$. So we get

$$a^1 = a^{0+1} = a^0 \cdot a = a$$

So it must be that $a^0 = 1$.

The first answer makes no sense because this would imply that

$$a^1 = a^{0+1} = a^0 \cdot a = 0 \cdot a = 0$$

whereas we know it should be a.

The third answer makes no sense because the answer is irrelevant with respect to the question.

The fourth answer makes no sense because it is a mathematical definition that the product of zero terms is equal to 1, just like the sum of zero terms is equal to 0.

4. The number of ways that one can divide 10 distinguishable objects into 3 indistinguishable *non-empty* piles, is:

$$\left\{ {10 \atop 3} \right\} = 9330$$

In how many different ways can one do this if the piles are *also* distinguishable?

(d) is correct

(a) $10^3 - \binom{3}{2}10^2 - \binom{3}{1}10^1 = 670$
(b) $3^{10} - \binom{3}{2}2^{10} - \binom{3}{1}1^{10} = 55974$
(c) $3 \cdot 9330 = 27990$
(d) $3! \cdot 9330 = 55980$

Answer (d) is correct. An algorithm to create such a distribution is by first dividing the objects over three indistinguishable non-empty piles, which can be done in 9330 ways. After that, we can put three different labels on the three piles, which can be done in $3! = 6$ ways. So in total there are $6 \cdot 9330$ ways to divide the objects over the piles.

It is clear that none of the other options can be correct at the same time.

4. The number of ways that one can divide 10 distinguishable objects in 3 indistinguishable *non-empty* piles, is:

$$\left\{ {10 \atop 3} \right\} = 9330$$

In how many different ways can one do this if the objects are *also* indistinguishable?

(a) is correct

(a) 8
(b) $\binom{10}{3} = 120$
(c) $9330/3 = 3110$
(d) $9330/10 = 933$

Answer (a) is correct. If both the objects and the piles are indistinguishable, a division is characterized solely by the number of objects in a pile. We can easily list these different possibilities systematically:

$$\begin{array}{ccccc} 8-1-1 & 7-2-1 & 6-3-1 & 5-4-1 & 4-4-2 \\ & & 6-2-2 & 5-3-2 & 4-3-3 \end{array}$$

Recall that the piles should not be empty. So we can do this in eight ways.

It is clear that none of the other options can be correct at the same time.

5. We want to translate the following English sentence to a modal formula:

> *Work from home, unless it is absolutely necessary that you go to work.*

Which of the following logics is the most suitable for this?

(a) Doxastic logic.

(b) is correct

(b) Deontic logic.
(c) Epistemic logic.
(d) Temporal logic.

Answer (b) is correct. The sentence can be interpreted as 'You must work from home, unless you really must go to work'. So therefore deontic logic.

The first answer makes no sense because doxastic logic is about belief, which is not appropriate in this situation.

The third answer makes no sense because epistemic logic is about knowledge, which is not appropriate in this situation.

The fourth answer makes no sense because temporal logic is about time, which is not appropriate in this situation.

5. We want to translate the following English sentence to a modal formula:

   An organization or business can be closed down for 14 days, if contract tracing by the Municipal Health Service (GGD) shows that an infection occurred there.

   Which of the following logics is the most suitable for this?

   If there are different modalities in the sentence, you should consider the outermost one. For instance 'You must never cheat on an exam' is deontic and not temporal.

(a) is correct

   (a) Alethic logic (the logic of necessity and possibility).
   (b) Doxastic logic.
   (c) Epistemic logic.
   (d) Temporal logic.

   Answer (a) is correct. The outer modality is in the 'can be closed down', which indicates a possibility. So alethic logic is the most suitable.

   The second answer makes no sense because doxastic logic is about belief, which is not appropriate in this situation.

   The third answer makes no sense because epistemic logic is about knowledge, which is not appropriate in this situation.

   The fourth answer makes no sense because temporal logic is about time, which is not appropriate in this situation, because that is part of the inner modality.

6. Which of the following LTL formulas does *not* hold in all LTL models?

   (a) $\mathcal{G}a \to \mathcal{G}\mathcal{G}a$
   (b) $\mathcal{G}a \to \mathcal{G}\mathcal{F}a$

(c) is correct

   (c) $\mathcal{F}a \to \mathcal{F}\mathcal{G}a$
   (d) $\mathcal{F}a \to \mathcal{F}\mathcal{F}a$

Answer (c) is correct. Note that $\mathcal{F}a \to \mathcal{F}\mathcal{G}a$ implies that if there will be a moment in the future (including now) that a holds, then there will also be a moment in the future (including now) that a always holds from that moment on. In the Kripke LTL model where a only holds now, but never again, this is not true.

The first answer makes no sense because it is true that if a always holds from now on, then it will also always hold from now on.

The second answer makes no sense because it is true that if a always holds from now on, then it will also always be the case that in the future (including the current) moment a will hold.

The fourth answer makes no sense because it is true that in the future (including now) a will hold, it is also true that in the future (including now) a will hold in the future (including now), because of the 'including now' allows to strip one of the $\mathcal{F}$'s.

6. Which of the following LTL formulas does *not* hold in all LTL models?

(a) is correct

(a) $\mathcal{X}a \to \mathcal{X}\mathcal{X}a$
(b) $\mathcal{X}a \to \mathcal{X}\mathcal{F}a$
(c) $\mathcal{X}a \to \mathcal{F}\mathcal{X}a$
(d) $\mathcal{X}a \to \mathcal{F}\mathcal{F}a$

Answer (a) is correct. The formula $\mathcal{X}a \to \mathcal{X}\mathcal{X}a$ implies that if a holds in the next moment, then it will also hold in the next moment after that moment. But if we take a Kripke LTL model where a is only true in the second moment, then $\mathcal{X}a$ does hold but $\mathcal{X}\mathcal{X}a$ does not.

The second answer makes no sense because it is true that if a holds in the next moment, then $\mathcal{F}a$ will also hold in the next moment, because $\mathcal{F}$ includes the current moment.

The third answer makes no sense because it is true that if a holds in the next moment, then in the future (including now) $\mathcal{X}a$ also holds, because $\mathcal{F}$ includes the current moment.

The fourth answer makes no sense because it is true that if a holds in the next moment, then $\mathcal{F}\mathcal{F}a$ also holds because if $\mathcal{X}a$ holds, we know that in the future a holds, and then because $\mathcal{F}$ includes the current moment, we may add a second $\mathcal{F}$ and it will still be true.

Open questions

7. Someone defines

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= 2a_n + n \qquad\qquad \text{for } n \geq 0 \end{aligned}$$

and then proves using induction that

$$a_{n-1} + n = 2^n \qquad\qquad \text{for } n \geq 1$$

The exercise is to write down the *induction hypothesis* for this proof, where the proof follows the induction scheme from our course.

After that, also explain why it is justified to *assume* this induction hypothesis in the proof by induction.

(Note that you do **not** need to write down the proof of the induction step, let alone the full induction proof. Only step six from the scheme is requested, with an explanation why this step is the way it is.)

Step 6 is: 'Assume that we already know that $P(k)$ holds, i.e. we assume that $a_{k-1} + k = 2^k$.'

This means that the induction hypothesis is the statement $a_{k-1} + k = 2^k$.

You may assume the induction hypothesis when proving the induction step for k, because it follows from the base case together with all the earlier instances of the induction step proof, i.e., the induction steps for $1, \ldots, k-1$.

The base case gives $P(1)$, the induction step for 1 uses that to prove $P(2)$, the induction step for 2 uses that to prove $P(3)$, and so on, until the induction step for $k-1$ uses that to prove $P(k)$. Therefore, at k we may use that we already know that $P(k)$ holds.

7. Someone wants to prove by induction:

$$\binom{n}{2} = \frac{1}{2}n(n-1) \qquad\qquad \text{for } n \geq 2$$

For this he or she wants to have the proof follow the recursive definition of Pascal's triangle.

The exercise is to write down the *base case* part of this induction proof.

After that, also give a sufficiently large part of Pascal's triangle, and indicate which number in Pascal's triangle is referenced from this base case.

(Note that you do **not** need to write down the full induction proof. Only steps three and four from the scheme are requested, with an explanation of the connection to Pascal's triangle.)

In the base case we have to prove that $\binom{2}{2} = \frac{1}{2} \cdot 2 \cdot (2-1)$. This holds because

$$\binom{2}{2} = 1 = (2-1) = 1 \cdot (2-1) = \frac{1}{2} \cdot 2 \cdot (2-1)$$

The start of Pascal's triangle is:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & \boxed{1} \end{array}$$

And the $\binom{2}{2}$ is marked.

8. Give a Kripke model that shows that:

$$\not\models \Box\Diamond a \to \Diamond\Box a$$

Explain your answer.

A model with a single world x_1 with no successors will do. The valuation does not matter.

$$\mathcal{M}_1 \quad := \; x_1 \; \bigcirc$$

If we want to prove that $\not\models \Box\Diamond a \to \Diamond\Box a$ holds, we have to show that there is at least one model $\mathcal{M}$ for which $\mathcal{M} \not\models \Box\Diamond a \to \Diamond\Box a$. In order to prove this, we have to show that there is at least one world x in the model $\mathcal{M}$ for which $x \nVdash \Box\Diamond a \to \Diamond\Box a$. In this case we take $\mathcal{M}_1$ for $\mathcal{M}$ and x_1 for x.

Now to prove that $x_1 \nVdash \Box\Diamond a \to \Diamond\Box a$, we show that $x_1 \Vdash \Box\Diamond a$ and $x_1 \nVdash \Diamond\Box a$.

The first claim holds by default, because any formula of the form $\Box f$ holds in x_1, due to the fact that x_1 has no successors. The second claim also holds by default, because any formula of the form $\Diamond f$ can only hold if there is at least one successor where f holds, but there is no successor at all in x_1, so certainly no successor where $\Box a$ holds.

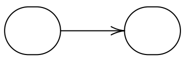
So $x_1 \nVdash \Box\Diamond a \to \Diamond\Box a$, so $\mathcal{M}_1 \not\models \Box\Diamond a \to \Diamond\Box a$, and hence $\not\models \Box\Diamond a \to \Diamond\Box a$.

8. Give a Kripke model that shows that:

$$\not\models \Diamond\Box a \to \Box\Diamond a$$

Explain your answer.

A model with two worlds x_1 and x_2 where x_2 is the only successor of x_1 and x_2 has no successors will do. The valuation does not matter.

$\mathcal{M}_1 \quad := x_1$  x_2

If we want to prove that $\not\models \Diamond\Box a \to \Box\Diamond a$ holds, we have to show that there is at least one model $\mathcal{M}$ for which $\mathcal{M} \not\models \Diamond\Box a \to \Box\Diamond a$. In order to prove this, we have to show that there is at least one world x in the model $\mathcal{M}$ for which $x \nVdash \Diamond\Box a \to \Box\Diamond a$. In this case we take $\mathcal{M}_1$ for $\mathcal{M}$ and x_1 for x.

Now to prove that $x_1 \nVdash \Diamond\Box a \to \Box\Diamond a$, we show that $x_1 \Vdash \Diamond\Box a$ and $x_1 \nVdash \Box\Diamond a$.

In order for the first claim to hold, there has to be a successor of x_1 where $\Box a$ holds. World x_2 is such a successor, because $x_2 \Vdash \Box a$ by definition, because any formula of the form $\Box f$ holds in x_2, due to the fact that x_2 has no successors. In order for the second claim to hold, there has to be a successor of x_1 where $\Diamond a$ does not hold. Since x_2 is the only successor of x_1, it has to be x_2. So we have to prove that $x_2 \nVdash \Diamond a$. However, this holds by default, because any formula of the form $\Diamond f$ can only hold if there is at least one successor where f holds, but there is no successor at all in x_2, so certainly no successor for which a holds.

So $x_1 \nVdash \Diamond\Box a \to \Box\Diamond a$, so $\mathcal{M}_1 \not\models \Diamond\Box a \to \Box\Diamond a$, and hence $\not\models \Diamond\Box a \to \Box\Diamond a$.