Formal Reasoning 2020
Solutions Test Blocks 1, 2 and 3: Additional Test
(16/12/20)

There are six multiple choice questions and two open questions. The open questions will be at the end of the test. Each multiple choice question is worth 10 points, and the open questions are worth 15 points. The first ten points are free. Good luck!

Multiple choice questions

1. The *exclusive or* operation in propositional logic, with symbol $\oplus$, is defined by the following truth table:

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

The formula $a \oplus b$ should be read as 'a, or b, but not both'.

Which of the following formulas is *not* logically equivalent to $a \oplus b$?

(b) is correct

(a) $a \wedge (\neg b) \vee (\neg a) \wedge b$

(b) $a \vee b \wedge \neg(a \wedge b)$

(c) $\neg a \leftrightarrow b$

(d) $\neg(a \leftrightarrow b)$

Answer (b) is correct. The short answer is that if $v(a) = 1$ and $v(b) = 1$, then $v(a \vee b \wedge \neg(a \wedge b)) = 1$, whereas $v(a \oplus b) = 0$. Note that the official notation of the formula is $(a \vee (b \wedge \neg(a \wedge b)))$ and not $((a \vee b) \wedge \neg(a \wedge b))$.

The long answer provides all truth tables and it is clear that all other formulas are indeed logically equivalent, because the tables are the same.

a	b	$a \oplus b$	$\neg a$	$\neg b$	$a \wedge (\neg b)$	$(\neg a) \wedge b$	$a \wedge (\neg b) \vee (\neg a) \wedge b$
0	0	0	1	1	0	0	0
0	1	1	1	0	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	0	0	0	0

a	b	$a \oplus b$	$a \wedge b$	$\neg(a \wedge b)$	$b \wedge \neg(a \wedge b)$	$a \vee b \wedge \neg(a \wedge b)$
0	0	0	0	1	0	0
0	1	1	0	1	1	1
1	0	1	0	1	0	1
1	1	0	1	0	0	1

a	b	$a \oplus b$	$\neg a$	$\neg a \leftrightarrow b$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	1
1	1	0	0	0

a	b	$a \oplus b$	$a \leftrightarrow b$	$\neg(a \leftrightarrow b)$
0	0	0	1	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	0

1. The *Sheffer stroke* operation in propositional logic, with symbol $|$, is defined by the following truth table:

a	b	$a \mid b$
0	0	1
0	1	1
1	0	1
1	1	0

This is also known as the *nand* operation, as it corresponds to the *nand*-gate which is one of the basic gates in logical circuits in a computer.

Which of the following formulas is *not* logically equivalent to $a \mid b$?

(a) $\neg(a \wedge b)$

(b) is correct

(b) $\neg a \wedge \neg b$

(c) $a \rightarrow \neg b$

(d) $b \rightarrow \neg a$

Answer (b) is correct. The short answer is that if $v(a) = 0$ and $v(b) = 1$, then $v(\neg a \wedge \neg b) = 0$, whereas $v(a \mid b) = 1$.

The long answer provides all truth tables and it is clear that all other formulas are indeed logically equivalent, because the tables are the same.

a	b	$a \mid b$	$a \wedge b$	$\neg(a \wedge b)$
0	0	1	0	1
0	1	1	0	1
1	0	1	0	1
1	1	0	1	0

a	b	$a \mid b$	$\neg a$	$\neg b$	$\neg a \wedge \neg b$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	1	0	1	0
1	1	0	1	0	0

a	b	$a \mid b$	$\neg b$	$a \rightarrow \neg b$
0	0	1	1	1
0	1	1	0	1
1	0	1	1	1
1	1	0	0	0

a	b	$a \mid b$	$\neg a$	$b \rightarrow \neg a$
0	0	1	1	1
0	1	1	1	1
1	0	1	0	1
1	1	0	0	0

2. Consider the model $M := (\mathbb{N}, 0, +)$ and the interpretation I defined by:

N	$\mathbb{N}$
o	0
$A(x, y, z)$	$x + y = z$

Which of the following statements does not hold?

(a) $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(o, x, y)$
(b) $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(x, o, y)$
(c) $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(y, o, x)$
(d) is correct (d) $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(x, y, o)$

Answer (d) is correct. The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(x, y, o)$ means that for each natural number x there is a natural number in y such that $x + y = 0$. However, this cannot hold. If $x = 37$ then y should be -37 in order to get $x + y = 0$, but $-37 \notin \mathbb{N}$. So it doesn't hold for all $x \in \mathbb{N}$.

The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(o, x, y)$ means that for each natural number x there exists a natural number y such that $0 + x = y$. This holds because we can take $y = x$.

The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(x, o, y)$ means that for each natural number x there exists a natural number y such that $x + 0 = y$. This holds because we can take $y = x$.

The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, A(y, o, x)$ means that for each natural number x there exists a natural number y such that $y + 0 = x$. This holds because we can take $y = x$.

2. Consider the model $M := (\mathbb{N}, -)$ and the interpretation I defined by:

N	$\mathbb{N}$
$S(x, y, z)$	$x - y = z$

Which of the following statements does not hold?

(a) is correct (a) $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(x, y, y)$
(b) $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(x, x, y)$
(c) $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(x, y, x)$
(d) $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(y, x, x)$

Answer (a) is correct. The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(x, y, y)$ means that for each natural number x there is a natural number y such that $x - y = y$. However, this doesn't hold if $x = 1$. Because $x - y$ needs to be a natural number, y can only be 0 or 1. If $y = 0$ we get $1 - 0 = 0$ and if $y = 1$ we get $1 - 1 = 1$. Obviously, in both cases the statement is not true. So the statement doesn't hold.

The statement $(M, I) \vDash \forall x \in N\, \exists y \in N\, S(x, x, y)$ means that for each natural number x there exists a natural number y such that $x - x = y$. This holds because we can always take $y = 0$.

The statement $(M, I) \models \forall x \in N\, \exists y \in N\, S(x, y, x)$ means that for each natural number x there exists a natural number y such that $x - y = x$. This holds because we can always take $y = 0$.

The statement $(M, I) \models \forall x \in N\, \exists y \in N\, S(y, x, x)$ means that for each natural number x there exists a natural number y such that $y - x = x$. This holds because we can always take $y = 2x$.

3. If for a language L is given that $L^* = L$, what does *not* necessarily follow?

   (a) $LL = L$

   (b) $\lambda \in L$

   (c) is correct — (c) L is infinite

   (d) L is non-empty

   Answer (c) is correct. If $L = \{\lambda\}$, then $L^* = L$. So the assumption that $L^* = L$ does not imply that L is infinite.

   Statement $LL = L$ holds, because if $w \in LL$, then $w = w_1 w_2 \in L^* = L$, and if $w \in L$, then $w = w\lambda \in LL$, because by definition $\lambda \in L^*$ and hence $\lambda \in L$.

   Statement $\lambda \in L$ holds, because by definition $\lambda \in L^*$, and hence $\lambda \in L$.

   Statement L is non-empty holds, because if $L = \emptyset$, then $L^* = \{\lambda\}$ which contradicts $L^* = L$.

3. If for a language L is given that $L^R = L$, what *does* necessarily follow?

   (a) is correct — (a) if $w \in L^*$ then also $w^R \in L^*$

   (b) there is a $w \in L$ for which also $w^R \in L$

   (c) $\lambda \in L$, because $\lambda^R = \lambda$

   (d) there is a $w \in L$ with $w^R = w$

   Answer (a) is correct. Statement 'if $w \in L^*$ then also $w^R \in L^*$' holds. Let $w \in L^*$. Then there exists $k \in \mathbb{N}$ such that $w = w_1 \cdots w_k$ and $w_i \in L$ for $i \in \{1, \ldots, k\}$. Note that $w^R = (w_1 \cdots w_k)^R = w_k^R \cdots w_1^R$. However, because $L^R = L$, we have that $w_i^R \in L$ for $i \in \{1, \ldots, k\}$. Hence $w^R = w_k^R \cdots w_1^R \in L^*$.

   Statement 'there is a $w \in L$ for which also $w^R \in L$' does not hold. Take $L = \emptyset$. Then $L^R = \emptyset = L$. However, there is no $w \in L$, so certainly no $w \in L$ for which $w^R \in L$.

   Statement '$\lambda \in L$, because $\lambda^R = \lambda$' does not hold. Take $L = \{a\}$. Then $L^R = L$, but $\lambda \notin L$.

   Statement 'there is a $w \in L$ with $w^R = w$' does not hold. Take $L = \{ab, ba\}$. Then $L^R = L$, but $(ab)^R = ba \neq ab$ and $(ba)^R = ab \neq ba$.

4. Let be given a deterministic finite automaton $M := \langle \Sigma, Q, q_0, F, \delta \rangle$, with $F \neq Q$. What do we know?

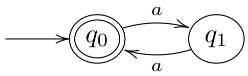
   (a) $\lambda \notin L(M)$

   (b) M has a sink

   (c) $q_0 \notin F$

(d) is correct (d) none of the above

Answer (d) is correct. Let

$M :=$ 

Then this automaton is a deterministic finite automaton with $F \neq Q$, where $\Sigma = \{a\}$.

However, because $q_0 \in F$ the empty word λ is accepted, so statement '$\lambda \notin L(M)$' does not hold.

However, M doesn't have a sink, so statement 'M has a sink' does not hold.

In addition $q_0 \in F$, so statement '$q_0 \notin F$' does not hold.

Hence the proper answer is 'none of the above'.

4. Let be given a deterministic finite automaton $M := \langle \Sigma, Q, q_0, F, \delta \rangle$, with $\lambda \notin L(M)$. What does *not* follow?

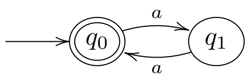
   (a) $F \neq Q$

(b) is correct (b) M has a sink

   (c) $q_0 \notin F$

   (d) none of the above

   Answer (b) is correct. Let

   $M :=$ 

   Then this automaton is a deterministic finite automaton with $\lambda \notin L(M)$, where $\Sigma = \{a\}$. However, M has no sink.

   Statement '$L(M) \neq \Sigma^*$' holds for any automaton such that $\lambda \notin L(M)$, because $\lambda \in \Sigma^*$ and $\lambda \notin L(M)$.

   Statement '$q_0 \notin F$' holds for any automaton such that $\lambda \notin L(M)$, because if $q_0 \in F$ then automatically $\lambda \in L(M)$.

5. The four color theorem says that planar graphs always have a chromatic number that is not higher than four. Is the converse (each graph with chromatic number not higher than four is always planar) also true?

(a) is correct (a) No, the graph $K_{3,3}$ is not planar, but it has chromatic number two.

   (b) Yes, the graph K_4 has chromatic number four, and is planar.

   (c) No, the graph K_4 has chromatic number four, but can be drawn with crossing edges.

   (d) Yes, the graph K_5 has chromatic number five, and is not planar.

Answer (a) is correct. Because $K_{3,3}$ is a bipartite graph, its chromatic number is at most two and it is not difficult to see that it is actually two. However, it is a well-known fact that $K_{3,3}$ and K_5 are non-planar. So we have an example of a graph with a chromatic number not higher than four which is not planar.

The second answer makes no sense because it only gives a single example of a graph with chromatic number not higher than four, but the claim is about all such graphs.

The third answer makes no sense because planarity means that the graph can be drawn without crossing edges, so the fact that K_4 can be drawn with crossing edges is irrelevant.

The fourth answer makes no sense because it gives an example with chromatic number higher than four, but the claim is about graphs with a chromatic number not higher than four.

5. Euler's theorem is stated in the course notes about connected graphs with at least two vertices. Are both of these conditions necessary?

   (a) Yes, because there cannot be an Eulerian path if the graph is not connected or has at most one vertex, no matter what the degrees are.

   (b) Yes, but for certain graphs there still can be an Eulerian path, even if the graph is not connected or has at most one vertex.

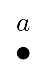
   (b) is correct

   (c) No, there are no connected graphs with less than two vertices, because then there cannot be a path in the graph, so the requirement on the number of vertices is not necessary.

   (d) No, all graphs with less than two vertices are connected, so the requirement on the number of vertices is not necessary.

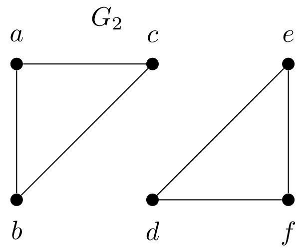
Answer (b) is correct. Let us see what happens if we drop the requirement on the number of vertices being at least two. Let $G_1 := \langle \{a\}, \emptyset \rangle$, hence a graph with only one vertex and no edges:

G_1



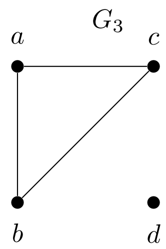
By definition it is connected. Then automatically this single vertex has degree zero, which is even. So in fact, all vertices have an even degree and an Eulerian path should exist, but since there is no path, there certainly is no Eulerian path. So the condition about the vertices is needed.

Now let us see what happens if we drop the requirement of being connected. Let $G_2 := \langle \{a, b, c, d, e, f\}, (a, b), (b, c), (c, a), (d, e), (e, f), (f, d) \rangle$:



Then G_2 is not connected, but all vertices have an even degree. So an Eulerian path should exist, but this cannot be the case since the graph has two cycles that are not connected. So the condition about being connected is also needed.

However, let $G_3 := \langle \{a, b, c, d\}, (a, b), (b, c), (c, a) \rangle$:

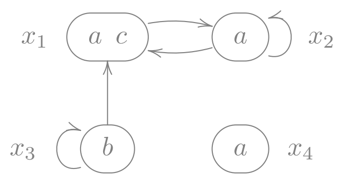


Then the graph is not connected, but there is an Eulerian path $a \to b \to c$. So for some situations, Eulerian paths may still exist.

The first answer makes no sense because we just presented an example of a graph that is not connected, but has an Eulerian path (and circuit).

The third and the fourth answer make no sense because the answer is 'yes'.

6. Consider the Kripke model $\mathcal{M}$:



In which worlds does the formula $\Box\Diamond c$ hold?

(a) x_1 and x_2

(b) is correct

(b) x_1 and x_4

(c) only x_1

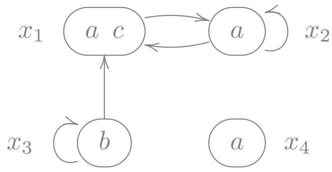
(d) none

Answer (b) is correct. Let us provide a table for the $\Vdash$ relation:

$\Vdash$	c	$\Diamond c$	$\Box\Diamond c$
x_1	1	0	1
x_2	0	1	0
x_3	0	1	0
x_4	0	0	1

So it follows from the table that $\Box\Diamond c$ holds in x_1 and x_4, but not in x_2 and x_3.

6. Consider the Kripke model $\mathcal{M}$:



In which worlds does the formula $\Diamond\Box c$ hold?

(a) x_1 and x_2

(b) x_1 and x_4

(c) only x_1

(d) is correct (d) none

Let us provide a table for the $\Vdash$ relation:

$\Vdash$	c	$\Box c$	$\Diamond\Box c$
x_1	1	0	0
x_2	0	0	0
x_3	0	1	0
x_4	1	1	0

So it follows from the table that there is no world where $\Diamond\Box c$ holds.

Open questions

7. Translate into a formula of predicate logic:

Grass is green, but grass is not the only green plant.

Use the dictionary:

P	the domain of plants
$G(x)$	x is a grass
$V(x)$	x is green

For instance:

$$\forall x \in P\, [G(x) \to V(x)] \land \exists x \in P\, [\neg G(x) \land V(x)]$$

7. Give a regular expression for the language:

$$\big\{ w \in \{a, b\}^* \mid w \text{ contains } ab\text{, but } w \text{ does not contain } ba \big\}$$

For instance:

$$a^*abb^*$$

7. Describe an LTL Kripke model in which the following LTL formula is true:

$$\neg(a\,\mathcal{U}\,b)$$

Truth in a model means truth in all worlds of the model, so we need to make sure the formula holds in all worlds.

Any model that has no b somewhere will do.

So take for instance: $\mathcal{M} := \langle W, R, V\rangle$ where

$$W = \{x_i \mid i \in \mathbb{N}\}$$

$$R(x_i) = \{x_j \mid j \in \mathbb{N} \text{ and } j \geq i\}$$
$$V(x_i) = \emptyset \quad \text{for } i \in \mathbb{N}$$

8. Write the following propositional formula according to the official grammar from the course notes, and give the full truth table:

$$\neg(\neg(((\neg a) \to a) \to a))$$

All we have to do is remove some parenthesis for the negation:

$$\neg\neg((\neg a \to a) \to a)$$

The full truth table is:

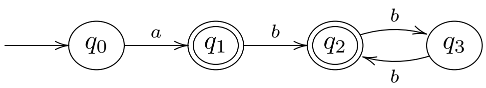
a	$\neg a$	$(\neg a \to a)$	$(\neg a \to a) \to a$	$\neg((\neg a \to a) \to a)$	$\neg\neg((\neg a \to a) \to a)$
0	1	0	1	0	1
1	0	1	1	0	1

8. Give a non-deterministic finite automaton for the language

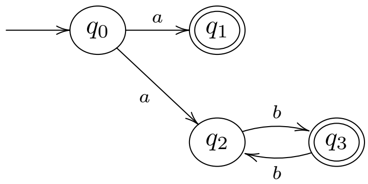
$$\{a\} \cup \{ab^n \mid n \text{ is odd}\}$$

with at most four states.

For instance:

$\mathcal{M} :=$ 

or

$\mathcal{M} :=$ 

8. We define recursively:

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= 2a_n - n \qquad\qquad \text{for } n \geq 0 \end{aligned}$$

Prove by induction that $a_n = n + 1$ for all $n \geq 0$.

0 **Proposition:**

$a_n = n + 1$ for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \quad := \quad a_n = n + 1$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that

$a_0 = 0 + 1$

4 This indeed holds, because

$$a_0 = 1 = 0 + 1$$

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that

$a_k = k + 1$ (Induction Hypothesis IH)

7 We now show that $P(k + 1)$ also holds, i.e. we show that

$a_{k+1} = k + 1 + 1$

8 This indeed holds, because

$$\begin{aligned} a_{k+1} &= 2a_k - k \\ &\overset{\text{IH}}{=} 2(k+1) - k \\ &= 2k + 2 - k \\ &= k + 2 \\ &= k + 1 + 1 \end{aligned}$$

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.