Formal Reasoning 2020
Test Block 3: Discrete Mathematics and Modal Logic
(9/12/20)

There are six multiple choice questions and two open questions. The open questions will be at the end of the test. Each multiple choice question is worth 10 points, and the open questions are worth 15 points. And the first ten points are free. Good luck!

Multiple choice questions

1. A non-empty tree with n vertices has always exactly $n-1$ edges. A *forest* is a graph that has no cycles. Therefore a forest is what you would expect: a graph in which each component is a tree.

   Which of the following formulas is the generalization of the edge count formula to forests?

   (a) A forest with n vertices and c components has $n - c$ edges.
   (b) A forest with n vertices and c components has $(n - 1) + c$ edges.
   (c) A forest with n vertices and c components has $n + c - 2$ edges.
   (d) A forest with n vertices has $n - 1$ edges; the number of components does not matter.

1. Which of the following statements correctly characterizes the number of edges in a non-empty connected graph, in the sense that the bounds hold and are optimal?

   (a) A non-empty connected graph with n vertices has at least $n-1$ edges and at most $\frac{1}{2}n(n-1)$ edges.
   (b) A non-empty connected graph with n vertices has at least n edges and at most $\frac{1}{2}n(n-1)$ edges.
   (c) A non-empty connected graph with n vertices has at least $n-1$ edges and at most $\frac{1}{2}n(n+1)$ edges.
   (d) A non-empty connected graph with n vertices has at least n edges and at most $\frac{1}{2}n(n+1)$ edges.

2. Two graphs G_1 and G_2 are isomorphic and G_1 is planar. Is G_2 then also always planar?

   (a) Yes, it does not matter for planarity what the labels of the vertices are.
   (b) Yes, because G_1 and G_2 have the same chromatic number.
   (c) No, if G_1 is drawn without crossing edges, then you don't know whether G_2 has crossing edges or not.
   (d) No, it depends on the graphs G_1 and G_2 whether this is the case.

2. How many isomorphisms are there from $K_{2,3}$ to itself?

   (a) Only one.

(b) Six.
(c) Twelve.
(d) None of the above.

3. We want to give a recursive definition of $n! = 1 \cdot 2 \cdot \ldots \cdot n$, using equations:

$$\begin{aligned} 0! &= \ldots \\ (n+1)! &= n! \cdot (n+1) \qquad\qquad \text{for } n \geq 0 \end{aligned}$$

For which value for $0!$ will this recursive definition work?

(a) $0! = 0$
(b) $0! = 1$
(c) It does not matter what value for $0!$ one takes, the values of $n!$ for $n \geq 0$ will not change because of that.
(d) This is not possible, because if one uses the recursive equation backwards to calculate $0!$ from $1!$, one is dividing by zero.

3. We want to give a recursive definition of a^n, using equations:

$$\begin{aligned} a^0 &= \ldots \\ a^{n+1} &= a^n \cdot a \qquad\qquad \text{for } n \geq 0 \end{aligned}$$

For which value for a^0 will this recursive definition work?

(a) $a^0 = 0$
(b) $a^0 = 1$
(c) It does not matter what value for a^0 one takes, the values of a^n for $n \geq 0$ will not change because of that.
(d) This is not possible, because a^0 is the multiplication of zero a's, and you cannot multiply anything when there are no a's to be multiplied.

4. The number of ways that one can divide 10 distinguishable objects into 3 indistinguishable *non-empty* piles, is:

$$\left\{ {10 \atop 3} \right\} = 9330$$

In how many different ways can one do this if the piles are *also* distinguishable?

(a) $10^3 - \binom{3}{2}10^2 - \binom{3}{1}10^1 = 670$
(b) $3^{10} - \binom{3}{2}2^{10} - \binom{3}{1}1^{10} = 55974$
(c) $3 \cdot 9330 = 27990$
(d) $3! \cdot 9330 = 55980$

4. The number of ways that one can divide 10 distinguishable objects in 3 indistinguishable *non-empty* piles, is:

$$\left\{ {10 \atop 3} \right\} = 9330$$

In how many different ways can one do this if the objects are *also* indistinguishable?

(a) 8
(b) $\binom{10}{3} = 120$
(c) $9330/3 = 3110$
(d) $9330/10 = 933$

5. We want to translate the following English sentence to a modal formula:

   Work from home, unless it is absolutely necessary that you go to work.

   Which of the following logics is the most suitable for this?

   (a) Doxastic logic.
   (b) Deontic logic.
   (c) Epistemic logic.
   (d) Temporal logic.

5. We want to translate the following English sentence to a modal formula:

   An organization or business can be closed down for 14 days, if contract tracing by the Municipal Health Service (GGD) shows that an infection occurred there.

   Which of the following logics is the most suitable for this?

   If there are different modalities in the sentence, you should consider the outermost one. For instance 'You must never cheat on an exam' is deontic and not temporal.

   (a) Alethic logic (the logic of necessity and possibility).
   (b) Doxastic logic.
   (c) Epistemic logic.
   (d) Temporal logic.

6. Which of the following LTL formulas does *not* hold in all LTL models?

   (a) $\mathcal{G}a \to \mathcal{G}\mathcal{G}a$
   (b) $\mathcal{G}a \to \mathcal{G}\mathcal{F}a$
   (c) $\mathcal{F}a \to \mathcal{F}\mathcal{G}a$
   (d) $\mathcal{F}a \to \mathcal{F}\mathcal{F}a$

6. Which of the following LTL formulas does *not* hold in all LTL models?

   (a) $\mathcal{X}a \to \mathcal{X}\mathcal{X}a$
   (b) $\mathcal{X}a \to \mathcal{X}\mathcal{F}a$
   (c) $\mathcal{X}a \to \mathcal{F}\mathcal{X}a$
   (d) $\mathcal{X}a \to \mathcal{F}\mathcal{F}a$

Open questions

7. Someone defines

$$
\begin{aligned}
a_0 &= 1 \\
a_{n+1} &= 2a_n + n \qquad\qquad \text{for } n \geq 0
\end{aligned}
$$

   and then proves using induction that

$$a_{n-1} + n = 2^n \qquad\qquad \text{for } n \geq 1$$

   The exercise is to write down the *induction hypothesis* for this proof, where the proof follows the induction scheme from our course.

   After that, also explain why it is justified to *assume* this induction hypothesis in the proof by induction.

   (Note that you do **not** need to write down the proof of the induction step, let alone the full induction proof. Only step six from the scheme is requested, with an explanation why this step is the way it is.)

7. Someone wants to prove by induction:

$$\binom{n}{2} = \frac{1}{2}n(n-1) \qquad\qquad \text{for } n \geq 2$$

   For this he or she wants to have the proof follow the recursive definition of Pascal's triangle.

   The exercise is to write down the *base case* part of this induction proof.

   After that, also give a sufficiently large part of Pascal's triangle, and indicate which number in Pascal's triangle is referenced from this base case.

   (Note that you do **not** need to write down the full induction proof. Only steps three and four from the scheme are requested, with an explanation of the connection to Pascal's triangle.)

8. Give a Kripke model that shows that:

$$\not\models \Box\Diamond a \to \Diamond\Box a$$

   Explain your answer.

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   Explain your answer.