Mix-automatic sequences

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Adding a finite string in front of a periodic sequence yields an *ultimately periodic sequence*, so is of the shape uv^ω for $u, v \in \Gamma^+$

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So $a \in \{0,1\}^{\mathbb{N}}$ is 2-automatic if $\{(i)_2 \mid a_i = 1\}$ is regular

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The set of strings with an odd number of ones is regular, so m is 2-automatic

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$a \in \Gamma^{\mathbb{N}}$ is 2-automatic if and only if there exists Δ, $f : \Delta \to \Delta^2$, $x \in \Delta$, $f(x) = xu$, $\tau : \Delta \to \Gamma$, $a = \tau(f^{\omega}(x))$

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In words: *morphic* with respect to a *2-uniform* morphism

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A sequence a is 2-automatic if and only if $K_2(a)$ is finite

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Mix-automatic sequences form a proper extension of the class of automatic sequences

They arise from a generalization of finite state automata where the input alphabet is state-dependent

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The seminar project also may involve observations on complexity of mix-automatic sequences