Proving termination using the dependency pair method and SAT

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Papers

- SAT Solving for Termination Analysis with Polynomial Interpretations
  Carsten Fuhs, Jürgen Giesl, Aart Middeldorp, Peter Schneider-Kamp, René Thiemann and Harald Zankl.
- Tyrolean termination tool: Techniques and features
  Nao Hirokawa and Aart Middeldorp.

Example Term Rewriting system

Set of terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$, signature $\mathcal{F} = \{\neg, \vee, \wedge\}$ and disjoint set of variables $\mathcal{V}$

1. $\neg\neg x \to x$
2. $\neg(x \vee y) \to \neg x \wedge \neg y$
3. $\neg(x \wedge y) \to \neg x \vee \neg y$
4. $x \wedge (y \vee z) \to (x \wedge y) \vee (x \wedge z)$
5. $(y \vee z) \wedge x \to (x \wedge y) \vee (x \wedge z)$
6. $(x \vee y) \vee z \to x \vee (y \vee z)$

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Can we prove termination?

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$\neg(x \wedge \neg(y \vee x))$

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$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x)$

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$$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x) \to^1 \neg x \vee (y \vee x)$$

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$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x) \to^1 \neg x \vee (y \vee x)$
$\neg(x \wedge \neg(y \vee x))$

Example Term Rewriting system

Set of terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$, signature $\mathcal{F} = \{\neg, \vee, \wedge\}$ and disjoint set of variables $\mathcal{V}$

1. $\neg\neg x \rightarrow x$
2. $\neg(x \vee y) \rightarrow \neg x \wedge \neg y$
3. $\neg(x \wedge y) \rightarrow \neg x \vee \neg y$
4. $x \wedge (y \vee z) \rightarrow (x \wedge y) \vee (x \wedge z)$
5. $(y \vee z) \wedge x \rightarrow (x \wedge y) \vee (x \wedge z)$
6. $(x \vee y) \vee z \rightarrow x \vee (y \vee z)$

$\neg(x \wedge \neg(y \vee x)) \rightarrow^3 \neg x \vee \neg\neg(y \vee x) \rightarrow^1 \neg x \vee (y \vee x)$

$\neg(x \wedge \neg(y \vee x)) \rightarrow^2 \neg(x \wedge (\neg y \wedge \neg x))$

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$\neg(x \wedge \neg(y \vee x)) \rightarrow^3 \neg x \vee \neg\neg(y \vee x) \rightarrow^1 \neg x \vee (y \vee x)$

$\neg(x \wedge \neg(y \vee x)) \rightarrow^2 \neg(x \wedge (\neg y \wedge \neg x)) \rightarrow^3 \neg x \vee \neg(\neg y \wedge \neg x)$

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$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x) \to^1 \neg x \vee (y \vee x)$
$\neg(x \wedge \neg(y \vee x)) \to^2 \neg(x \wedge (\neg y \wedge \neg x)) \to^3 \neg x \vee \neg(\neg y \wedge \neg x)$
$\to^3 \neg x \vee (\neg\neg y \vee \neg\neg x)$

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$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x) \to^1 \neg x \vee (y \vee x)$
$\neg(x \wedge \neg(y \vee x)) \to^2 \neg(x \wedge (\neg y \wedge \neg x)) \to^3 \neg x \vee \neg(\neg y \wedge \neg x)$
$\to^3 \neg x \vee (\neg\neg y \vee \neg\neg x) \to^1 \neg x \vee (y \vee \neg\neg x)$

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6. $(x \vee y) \vee z \to x \vee (y \vee z)$

$$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x) \to^1 \neg x \vee (y \vee x)$$

$$\neg(x \wedge \neg(y \vee x)) \to^2 \neg(x \wedge (\neg y \wedge \neg x)) \to^3 \neg x \vee \neg(\neg y \wedge \neg x) \to^3 \neg x \vee (\neg\neg y \vee \neg\neg x) \to^1 \neg x \vee (y \vee \neg\neg x) \to^1 \neg x \vee (y \vee x)$$

Substitutions

$\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$

3 $\neg(x \wedge y) \to \neg x \vee \neg y$

$\sigma(x) = x,\ \sigma(y) = \neg(y \vee x)$

$\neg(x \wedge \neg(y \vee x)) \to^3 \neg x \vee \neg\neg(y \vee x)$

Minimal non-terminating terms

If a TRS $\mathcal{R}$ is not terminating, it has non-terminating terms and also minimal non-terminating terms.
$\mathcal{T}_\infty = \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid t$ is non-terminating and every proper subterm is terminating $\}$

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- $f(x, y) \to f(g(y), x)$
- $g(x) \to h(x)$

$f(f(x, z), y)$
$f(x, g(x))$

Minimal non-terminating terms

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Lemma. For every term $t \in \mathcal{T}_\infty$ there exist a rewrite rule $l \to r$, a substitution σ, and a non-variable subterm u of r such that $t \xrightarrow{>\epsilon}{}^{*} l\sigma \xrightarrow{\epsilon} r\sigma \trianglerighteq u\sigma$ and $u\sigma \in \mathcal{T}_\infty$

Dependency pairs

- $f(x, y) \to f(g(y), x)$
- $g(x) \to h(x)$

Dependency pairs

- $f(x, y) \rightarrow f(g(y), x)$
- $g(x) \rightarrow h(x)$
- $f^{\#}(x, y) \rightarrow f^{\#}(g(y), x)$
- $f^{\#}(x, y) \rightarrow g^{\#}(y)$

Dependency pairs

If $s = f(s_1, \ldots, s_n)$ then denote $s^\#$ as $f^\#(s_1, \ldots, s_n)$ If $l \to r \in \mathcal{R}$ and u is a subterm of r with defined root symbol such that u is not a proper subterm of l, then the rewrite rule $l^\# \to u^\#$ is called a dependency pair of $\mathcal{R}$.

The set of all dependency pairs of $\mathcal{R}$ is denoted by $DP(\mathcal{R})$

Dependency pairs

- $\neg^\#(x \vee y) \to \neg x \wedge^\# \neg y$
- $\neg^\#(x \vee y) \to \neg^\# x$
- $\neg^\#(x \vee y) \to \neg^\# y$
- $\neg^\#(x \wedge y) \to \neg x \vee^\# \neg y$
- $\neg^\#(x \wedge y) \to \neg^\# x$
- $\neg^\#(x \wedge y) \to \neg^\# y$
- $x \wedge^\# (y \vee z) \to (x \wedge y) \vee^\# (x \wedge z)$
- $x \wedge^\# (y \vee z) \to x \wedge^\# y$
- $x \wedge^\# (y \vee z) \to x \wedge^\# z$
- $(y \vee z) \wedge^\# x \to (x \wedge y) \vee^\# (x \wedge z)$
- $(y \vee z) \wedge^\# x \to x \wedge^\# y$
- $(y \vee z) \wedge^\# x \to x \wedge^\# z$
- $(x \vee y) \vee^\# z \to x \vee^\# (y \vee z)$
- $(x \vee y) \vee^\# z \to y \vee^\# z$

Dependency pairs

- For all $s \in \mathcal{T}_\infty$ there exist terms $t, u \in \mathcal{T}_\infty$ such that $s^\# \to_{\mathcal{R}}^* t^\# \to_{DP(\mathcal{R})} u^\#$

Dependency pairs

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- If $s = f(s_1, \ldots, s_n)$ then denote $s^\#$ as $f^\#(s_1, \ldots, s_n)$

Dependency pairs

- For all $s \in \mathcal{T}_\infty$ there exist terms $t, u \in \mathcal{T}_\infty$ such that $s^\# \to_{\mathcal{R}}^* t^\# \to_{DP(\mathcal{R})} u^\#$
- If $s = f(s_1, \ldots, s_n)$ then denote $s^\#$ as $f^\#(s_1, \ldots, s_n)$
- Every non-terminating TRS $\mathcal{R}$ admits an infinite rewrite sequence of the form
  $t_1 \to_{\mathcal{R}}^* t_2 \to_{DP(\mathcal{R})} t_3 \to_{\mathcal{R}}^* t_4 \to_{DP(\mathcal{R})} \cdots$

Dependency pairs

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- Reduced termination problem to proving that there cannot be infinitely many DP($\mathcal{R}$) steps

Dependency pairs

Note that for instance DP rule 10 can only be followed (directly) by DP rule 19 and 20

7 $\neg^\#(x \vee y) \to \neg x \wedge^\# \neg y$

8 $\neg^\#(x \vee y) \to \neg^\# x$

9 $\neg^\#(x \vee y) \to \neg^\# y$

10 $\neg^\#(x \wedge y) \to \neg x \vee^\# \neg y$

11 $\neg^\#(x \wedge y) \to \neg^\# x$

12 $\neg^\#(x \wedge y) \to \neg^\# y$

13 $x \wedge^\# (y \vee z) \to (x \wedge y) \vee^\# (x \wedge z)$

14 $x \wedge^\# (y \vee z) \to x \wedge^\# y$

15 $x \wedge^\# (y \vee z) \to x \wedge^\# z$

16 $(y \vee z) \wedge^\# x \to (x \wedge y) \vee^\# (x \wedge z)$

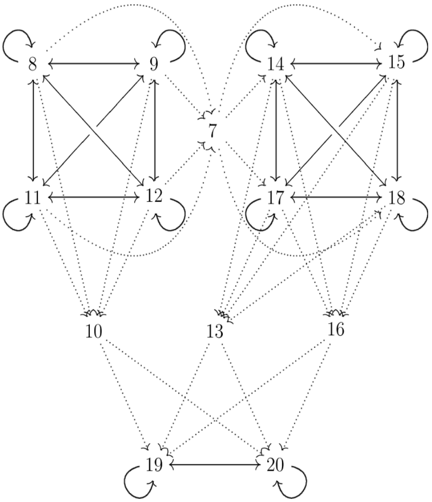
17 $(y \vee z) \wedge^\# x \to x \wedge^\# y$

18 $(y \vee z) \wedge^\# x \to x \wedge^\# z$

19 $(x \vee y) \vee^\# z \to x \vee^\# (y \vee z)$

20 $(x \vee y) \vee^\# z \to y \vee^\# z$

Dependency graph



7. $\neg^\#(x \vee y) \rightarrow \neg x \wedge^\# \neg y$
8. $\neg^\#(x \vee y) \rightarrow \neg^\# x$
9. $\neg^\#(x \vee y) \rightarrow \neg^\# y$
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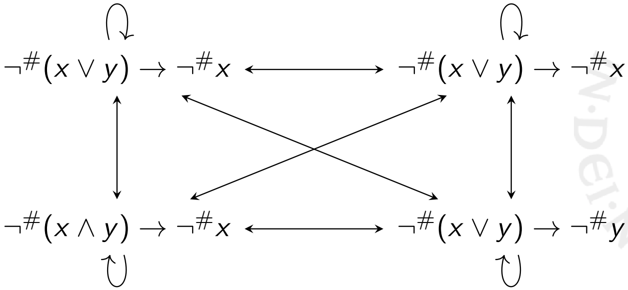
12. $\neg^\#(x \wedge y) \rightarrow \neg^\# y$
13. $x \wedge^\# (y \vee z) \rightarrow (x \wedge y) \vee^\# (x \wedge z)$
14. $x \wedge^\# (y \vee z) \rightarrow x \wedge^\# y$
15. $x \wedge^\# (y \vee z) \rightarrow x \wedge^\# z$
16. $(y \vee z) \wedge^\# x \rightarrow (x \wedge y) \vee^\# (x \wedge z)$
17. $(y \vee z) \wedge^\# x \rightarrow x \wedge^\# y$
18. $(y \vee z) \wedge^\# x \rightarrow x \wedge^\# z$
19. $(x \vee y) \vee^\# z \rightarrow x \vee^\# (y \vee z)$
20. $(x \vee y) \vee^\# z \rightarrow y \vee^\# z$

Dependency graph

Cycle of earlier example:
$\mathcal{C} = \{\neg^{\#}(x \vee y) \to \neg^{\#}x, \neg^{\#}(x \vee y) \to \neg^{\#}x,$
$\neg^{\#}(x \wedge y) \to \neg^{\#}x, \neg^{\#}(x \vee y) \to \neg^{\#}y\}$



Cycles in Dependency graph

Theorem For every non-terminating TRS $\mathcal{R}$ there exist a cycle C in DG($\mathcal{R}$) and an infinite rewrite sequence in $\mathcal{R} \cup \mathcal{C}$ of the form
$\mathcal{T}^{\#}_{\infty} \ni t_1 \rightarrow^{*}_{\mathcal{R}} t_2 \rightarrow_{\mathcal{C}} t_3 \rightarrow^{*}_{\mathcal{R}} t_4 \rightarrow_{\mathcal{C}} \ldots$
in which all rules of $\mathcal{C}$ are applied infinitely often.

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$\mathcal{T}_{\infty}^{\#} \ni t_1 \rightarrow_{\mathcal{R}}^{*} t_2 \rightarrow_{\mathcal{C}} t_3 \rightarrow_{\mathcal{R}}^{*} t_4 \rightarrow_{\mathcal{C}} \dots$
in which all rules of $\mathcal{C}$ are applied infinitely often.
We call such an infinite rewrite sequence $\mathcal{C}$-minimal

Using projections

- $f^{\#}(s(x), y) \to g^{\#}(y, x)$
- $g^{\#}(y, x) \to f^{\#}(x, y)$

Using projections

- $f^{\#}(s(x), y) \to g^{\#}(y, x)$
- $g^{\#}(y, x) \to f^{\#}(x, y)$
- $\pi(f) = 1$ and $\pi(g) = 2$
- $\pi(f(s(x), y)) = s(x) \triangleright x = \pi(g(y, x))$
- $\pi(g(y, x)) = x \trianglerighteq x = \pi(f(x, y))$

Using projections

- $f^{\#}(s(x), y) \rightarrow g^{\#}(y, x)$
- $g^{\#}(y, x) \rightarrow f^{\#}(x, y)$
- $\pi(f) = 1$ and $\pi(g) = 2$
- $\pi(f(s(x), y)) = s(x) \rhd x = \pi(g(y, x))$
- $\pi(g(y, x)) = x \trianglerighteq x = \pi(f(x, y))$

Note that $\rhd$ is well-founded
(There are no infinite sequences of the form $t_1 \rhd t_2 \rhd t_3 \rhd \dots$)

Using projections

$$\pi(\mathcal{C}) = \{\pi(l) \to \pi(r) \mid l \to r \in \mathcal{C}\}$$

Theorem Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ be a cycle in the DG($\mathcal{R}$). If there exists a simple projection π for $\mathcal{C}$ such that $\pi(\mathcal{C}) \subseteq \trianglerighteq$ and $\pi(\mathcal{C}) \cap \triangleright \neq \emptyset$ then there are no $\mathcal{C}$-minimal rewrite sequences

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Reminder: In a $\mathcal{C}$-minimal sequence, every rules of $\mathcal{C}$ is applied infinitely often.

Using projections

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Theorem Let $\mathcal{R}$ be a TRS and let $\mathcal{C}$ be a cycle in the DG($\mathcal{R}$). If there exists a simple projection π for $\mathcal{C}$ such that $\pi(\mathcal{C}) \subseteq \trianglerighteq$ and $\pi(\mathcal{C}) \cap \triangleright \neq \emptyset$ then there are no $\mathcal{C}$-minimal rewrite sequences

Reminder: In a $\mathcal{C}$-minimal sequence, every rules of $\mathcal{C}$ is applied infinitely often.

$$\pi(t_1) \trianglerighteq^* \pi(t_n) \triangleright \pi(t_{n+1}) \trianglerighteq^* \pi(t_m) \triangleright \pi(t_{m+1}) \trianglerighteq^* \ldots$$

Using projections

$$\mathcal{C}_1 = \{\neg^{\#}(x \vee y) \to \neg^{\#} x, \neg^{\#}(x \vee y) \to \neg^{\#} y, \neg^{\#}(x \wedge y) \to \neg^{\#} x, \neg^{\#}(x \wedge y) \to \neg^{\#} y\}$$

Using projections

$\mathcal{C}_1 = \{\neg^\#(x\vee y) \to \neg^\# x, \neg^\#(x\vee y) \to \neg^\# y,$
$\neg^\#(x\wedge y) \to \neg^\# x, \neg^\#(x\wedge y) \to \neg^\# y\}$

Choose $\pi_1(\neg^\#) = 1$

- $(x\vee y) \to x$
- $(x\vee y) \to x$
- $(x\wedge y) \to x$
- $(x\vee y) \to y$

Using projections

$\mathcal{C}_1 = \{\neg^\#(x \vee y) \to \neg^\# x, \neg^\#(x \vee y) \to \neg^\# y,$
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Choose $\pi_1(\neg^\#) = 1$

- $(x \vee y) \to x$
- $(x \vee y) \to x$
- $(x \wedge y) \to x$
- $(x \vee y) \to y$

$\pi_1(\mathcal{C}_1) \subseteq \triangleright$, thus this cycle $\mathcal{C}_1$ can be ignored.

Using projections

$$\mathcal{C}_2 = \{(x \vee y) \vee^{\#} z \rightarrow x \vee^{\#} (y \vee z),$$
$$(x \vee y) \vee^{\#} z \rightarrow y \vee^{\#} z\}$$

Using projections

$\mathcal{C}_2 = \{(x \vee y) \vee^{\#} z \to x \vee^{\#} (y \vee z),$
$(x \vee y) \vee^{\#} z \to y \vee^{\#} z\}$
Choose $\pi_2(\vee^{\#}) = 1$

- $x \vee y \to x$
- $x \vee y \to y$

Using projections

$\mathcal{C}_2 = \{(x \vee y) \vee^{\#} z \to x \vee^{\#} (y \vee z),$
$(x \vee y) \vee^{\#} z \to y \vee^{\#} z\}$
Choose $\pi_2(\vee^{\#}) = 1$

- $x \vee y \to x$
- $x \vee y \to y$

$\pi_2(\mathcal{C}_2) \subseteq \rhd$, thus this cycle can also be ignored.

Using projections

$$\mathcal{C}_3 = \{(y \vee z) \wedge^{\#} x \to x \wedge^{\#} y,$$
$$(y \vee z) \wedge^{\#} x \to x \wedge^{\#} z\}$$

Using projections

$\mathcal{C}_3 = \{(y \vee z) \wedge^{\#} x \to x \wedge^{\#} y,$
$(y \vee z) \wedge^{\#} x \to x \wedge^{\#} z\}$
Choosing $\pi(\wedge^{\#}) = 1$ gives

- $y \vee z \to x$

Using projections

$\mathcal{C}_3 = \{(y \vee z) \wedge^{\#} x \rightarrow x \wedge^{\#} y,$
$(y \vee z) \wedge^{\#} x \rightarrow x \wedge^{\#} z\}$
Choosing $\pi(\wedge^{\#}) = 1$ gives

- $y \vee z \rightarrow x$

And choosing $\pi(\wedge^{\#}) = 2$ gives

- $x \rightarrow y$
- $x \rightarrow z$

Using projections

$\mathcal{C}_3 = \{(y \vee z) \wedge^{\#} x \to x \wedge^{\#} y,$
$(y \vee z) \wedge^{\#} x \to x \wedge^{\#} z\}$
Choosing $\pi(\wedge^{\#}) = 1$ gives

- $y \vee z \to x$

And choosing $\pi(\wedge^{\#}) = 2$ gives

- $x \to y$
- $x \to z$

These are not compatible with $\trianglerighteq$

Looking at SCCs

- Problem: there can be exponentially many cycles in the dependency graph (in the number of dependency pairs)

Looking at SCCs

- Problem: there can be exponentially many cycles in the dependency graph (in the number of dependency pairs)
- solve strongly connected components (linear algorithm)

Looking at SCCs

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Looking at SCCs

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- solve strongly connected components (linear algorithm)
- If all pairs in in SCC are compatible with $\trianglerighteq$ after applying a simple projection, remove the ones that are compatible with $\triangleright$.
- Compute new SCCs among the remaining pairs

Using projections and SCCs

1. $\text{intlist}([]) \to []$
2. $\text{intlist}(x{:}y) \to s(x){:}\ \text{intlist}(y)$
3. $\text{int}(0, 0) \to 0{:}[]$
4. $\text{int}(0, s(y)) \to 0{:}\ \text{int}(s(0), s(y))$
5. $\text{int}(s(x), 0) \to []$
6. $\text{int}(s(x), s(y)) \to \text{intlist}(\text{int}(x, y))$

The term $\text{int}(s^m(0), s^n(0))$ evaluates to the list $[s^m(0), s^{m+1}(0), \ldots, s^n(0)]$;

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3. $\mathsf{int}(0, 0) \to 0{:}[\,]$
4. $\mathsf{int}(0, s(y)) \to 0{:}\ \mathsf{int}(s(0), s(y))$
5. $\mathsf{int}(s(x), 0) \to [\,]$
6. $\mathsf{int}(s(x), s(y)) \to \mathsf{intlist}(\mathsf{int}(x, y))$
7. $\mathsf{intlist}^{\#}(x : y) \to \mathsf{intlist}^{\#}(y)$
8. $\mathsf{int}^{\#}(0, s(y)) \to \mathsf{int}^{\#}(s(0), s(y))$
9. $\mathsf{int}^{\#}(s(x), s(y)) \to \mathsf{intlist}^{\#}\ (\mathsf{int}(x, y))$
10. $\mathsf{int}^{\#}(s(x), s(y)) \to \mathsf{int}^{\#}(x, y)$

Using projections and SCCs

7. $\text{intlist}^{\#}(x{:}y) \rightarrow \text{intlist}^{\#}(y)$
8. $\text{int}^{\#}(0, s(y)) \rightarrow \text{int}^{\#}(s(0), s(y))$
9. $\text{int}^{\#}(s(x), s(y)) \rightarrow \text{intlist}^{\#}(\text{int}(x, y))$
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Using projections and SCCs

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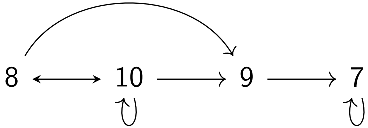

Using projections and SCCs

7. $\text{intlist}^{\#}(x{:}y) \to \text{intlist}^{\#}(y)$
8. $\text{int}^{\#}(0, s(y)) \to \text{int}^{\#}(s(0), s(y))$
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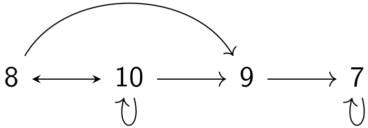


Contains two SCCs: $\{7\}$ and $\{8, 10\}$

Using projections and SCCs

7. $\mathsf{intlist}^{\#}(x{:}y) \to \mathsf{intlist}^{\#}(y)$

$\{7\}$ is handled by $\pi(\mathsf{intlist}^{\#}) = 1$:

- $x{:}y \to y$

Using projections and SCCs

7 $\mathsf{intlist}^\#(x{:}y) \to \mathsf{intlist}^\#(y)$

$\{7\}$ is handled by $\pi(\mathsf{intlist}^\#) = 1$:

- $x{:}y \to y$

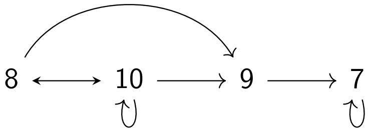
8 $\mathsf{int}^\#(0, s(y)) \to \mathsf{int}^\#(s(0), s(y))$

10 $\mathsf{int}^\#(s(x), s(y)) \to \mathsf{int}^\#(x, y)$

$\{8{,}10\}$ is handled by $\pi(\mathsf{int}^\#) = 2$

- $s(y) \to s(y)$
- $s(y) \to y$

Using projections and SCCs



After removing, 7 and 10, we are only left with 8.

Returning to dependency pairs

- Possible recursive calls

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- $t_1 \to_{\mathcal{R}}^{*} t_2 \to_{DP(\mathcal{R})} t_3 \to_{\mathcal{R}}^{*} t_4 \to_{DP(\mathcal{R})} \cdots$

Returning to dependency pairs

- Possible recursive calls
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Returning to dependency pairs

- Possible recursive calls
- $t_1 \rightarrow^*_{\mathcal{R}} t_2 \rightarrow_{DP(\mathcal{R})} t_3 \rightarrow^*_{\mathcal{R}} t_4 \rightarrow_{DP(\mathcal{R})} \cdots$
- Towards automation
- $\bigwedge_{u \rightarrow v \in DP(\mathcal{R})} u \succ v \wedge \bigwedge_{l \rightarrow r \in \mathcal{R}} l \succsim r$

Dependency pairs

1. $\text{half}(0) \rightarrow 0$
2. $\text{half}(s(0)) \rightarrow 0$
3. $\text{half}(s(s(x))) \rightarrow s(\text{half}(x))$
4. $\text{bits}(0) \rightarrow 0$
5. $\text{bits}(s(0)) \rightarrow s(0)$
6. $\text{bits}(s(s(x))) \rightarrow s(\text{bits}(s(\text{half}(x))))$

Dependency pairs

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5. $\mathsf{bits}(\mathsf{s}(0)) \to \mathsf{s}(0)$
6. $\mathsf{bits}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(x))))$
7. $\mathsf{half}^{\#}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{half}^{\#}(x)$
8. $\mathsf{bits}^{\#}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{half}^{\#}(x))$
9. $\mathsf{bits}^{\#}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{bits}^{\#}(\mathsf{s}(\mathsf{half}(x)))$

Polynomial interpretations

- Polynomial interpretation $\mathcal{P}ol$ maps symbols to polynomials with coefficients in $\mathbb{N}$

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- Polynomial interpretation $\mathcal{P}ol$ maps symbols to polynomials with coefficients in $\mathbb{N}$
- $[x]_{\mathcal{P}ol} = x$ and $[f(t_1, \ldots, t_n)] = [f]_{\mathcal{P}ol}([t_1]_{\mathcal{P}ol}, \ldots, [t_n]_{\mathcal{P}ol})$

Example polynomial interpretation

$\mathsf{half}_{\mathcal{P}ol} = \mathsf{half}^{\#}_{\mathcal{P}ol} = x_1$, $\mathsf{bits}_{\mathcal{P}ol} = \mathsf{bits}^{\#}_{\mathcal{P}ol} = \mathsf{s}_{\mathcal{P}ol} = x_1 + 1$, $0_{\mathcal{P}ol} = 0$.

Example polynomial interpretation

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1. $[\text{half}(0)] = 0 \geq 0 = [0]$
2. $[\text{half}(\text{s}(0))] = 1 \geq 0 = [0]$
3. $[\text{half}(\text{s}(\text{s}(\text{x})))] = x + 2 \geq x + 1 = [\text{s}(\text{half}(\text{x}))]$
4. $[\text{bits}(0)] = 1 \geq 0 = [0]$
5. $[\text{bits}(\text{s}(0))] = 2 \geq 1 = [\text{s}(0)]$
6. $[\text{bits}(\text{s}(\text{s}(\text{x})))] = x + 3 \geq x + 3 = [\text{s}(\text{bits}(\text{s}(\text{half}(\text{x}))))]$
7. $[\text{half}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 2 > x = [\text{half}^{\#}(\text{x})]$
8. $[\text{bits}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 3 > x = [\text{half}^{\#}(\text{x}))]$
9. $[\text{bits}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 3 > x + 2 = [\text{bits}^{\#}(\text{s}(\text{half}(\text{x})))]$

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7. $[\text{half}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 2 > x = [\text{half}^{\#}(\text{x})]$
8. $[\text{bits}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 3 > x = [\text{half}^{\#}(\text{x}))]$
9. $[\text{bits}^{\#}(\text{s}(\text{s}(\text{x})))] = x + 3 > x + 2 = [\text{bits}^{\#}(\text{s}(\text{half}(\text{x})))]$

Thus all rules are weakly decreasing and the DP's are strictly decreasing

Automating polynomial interpretations

$$t_1 \to_{\mathcal{R}}^* t_2 \to_{DP(\mathcal{R})} t_3 \to_{\mathcal{R}}^* t_4 \to_{DP(\mathcal{R})} \cdots \implies$$
$$t_1 \geq^* t_2 > t_3 \geq^* t_4 > \ldots$$

Automating polynomial interpretations

$[f(x_1, \ldots, x_n)] = a_0 + a_1 x^{e11} \cdots x^{e1n} + \cdots + a_m x^{em1} \cdots x^{emn}$
$\text{half}_{\mathcal{P}ol} = a x_1 + b$, $\text{s}_{\mathcal{P}ol} = c x_1 + d$
We can now transform the rule half(s(s(x)))→s(half(x)) to

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We can now transform the rule half(s(s(x)))→s(half(x)) to

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- $ac^2x + acd + ad + b - cax - cb - d \geq 0$

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- $ac^2x + acd + ad + b \geq cax + cb + d$
- $ac^2x + acd + ad + b - cax - cb - d \geq 0$
- $(ac^2 - ca)x + acd + ad + b - cb - d \geq 0$

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- $ac^2x + acd + ad + b \geq cax + cb + d$
- $ac^2x + acd + ad + b - cax - cb - d \geq 0$
- $(ac^2 - ca)x + acd + ad + b - cb - d \geq 0$
- $p_1x_1 + p_0 \geq 0$, where
  $p_1 = ac^2 - ca$ and $p_0 = acd + ad + b - cb - d$

Automating polynomial interpretations

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- $p_1 x_1 + p_0 \geq 0$, where
  $p_1 = ac^2 - ca$ and $p_0 = acd + ad + b - cb - d$
- $p_1 \geq 0 \wedge p_0 \geq 0$

Automating polynomial interpretations

- $\alpha_{p>0} = (p_0 > 0 \wedge p_1 \geq 0 \wedge \cdots \wedge p_k \geq 0)$
- $\alpha_{p\geq 0} = (p_0 \geq 0 \wedge p_1 \geq 0 \wedge \cdots \wedge p_k \geq 0)$
- $\alpha_{p=0} = (p_0 = 0 \wedge p_1 = 0 \wedge \cdots \wedge p_k = 0)$

Algorithm so far

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- Transform constraints to contain only $=$, $>$ and no subtractions
- Check satisfiability of Diophantine constraint

Encoding in SAT

$$\mathcal{J}(\langle \varphi_1, \ldots, \varphi_n \rangle) = 2^{n-1} \cdot \mathcal{J}(\varphi_1) + \cdots + 2 \cdot \mathcal{J}(\varphi_2) + \mathcal{J}(\varphi_n)$$

$\|p + q\| = B^+(\|p\|, \|q\|)$ and $\|p \cdot q\| = B^*(\|p\|, \|q\|)$

Encoding in SAT

- $B^+(\langle \varphi_1, \dots \varphi_n \rangle, \langle \psi_1, \dots, \psi_m \rangle) =$
  $B^+(\langle \varphi_1, \dots \varphi_n \rangle, \langle 0, \dots, 0, \psi_1, \dots, \psi_m \rangle)$ if $n > m$
- $B^+(\langle \varphi_1, \dots \varphi_n \rangle, \langle \psi_1, \dots, \psi_m \rangle) =$
  $B^+(\langle 0, \dots, 0, \varphi_1, \dots \varphi_n \rangle, \langle \psi_1, \dots, \psi_m \rangle)$ if $n < m$
- $B^+(\langle \varphi \rangle, \langle \psi \rangle) = \langle \varphi \wedge \psi, \varphi \oplus \psi \rangle$
- $B^+(\langle \varphi_1, \dots \varphi_n \rangle, \langle \psi_1, \dots, \psi_m \rangle) =$
  $\langle B^{2or3}(\varphi_1, \psi_1, \xi_1), B^{1or3}(\varphi_1, \psi_1, \xi_1), \xi_2, \dots, \xi_n \rangle$ if
  $B^+(\langle \varphi_2, \dots \varphi_n \rangle, \langle \psi_2, \dots, \psi_m \rangle) = \langle \xi_2, \dots \xi_n \rangle$

Encoding in SAT

$\|p > q\| = B^{>}(\|p\|, \|q\|)$ and $\|p = q\| = B^{=}(\|p\|, \|q\|)$

- $B^{=}(\langle \varphi_1, \ldots, _n \rangle, \langle \psi_1, \ldots, \psi_n \rangle) = (\varphi_1 \leftrightarrow \psi_1) \wedge \cdots \wedge (\varphi_n \leftrightarrow \psi_n)$
- $B^{>}(\langle \varphi \rangle, \langle \psi \rangle) = \varphi \wedge \neg \psi$
- $B^{>}(\langle \varphi_1, \ldots, _n \rangle, \langle \psi_1, \ldots, \psi_n \rangle) =$

$(\varphi_1 \wedge \neg \psi_1) \vee ((\varphi_1 \leftrightarrow \psi_1) \wedge B^{>}(\langle \varphi_2, \ldots, \varphi_n \rangle, \langle \psi_2, \ldots, \psi_n \rangle))$

Encoding in SAT

- $\{0, \ldots 2^k - 1\}$
- $\alpha \in \mathcal{C}$ and every number in $\alpha \leq 2^k - 1$. Then the size of $\|\alpha\|$ is in $\mathcal{O}(|\alpha|^2 \cdot k^2)$

Negative coefficients

1. $\mathsf{half}(0) \to 0$
2. $\mathsf{half}(\mathsf{s}(0)) \to 0$
3. $\mathsf{half}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{half}(x))$
4. $\mathsf{bits}(0) \to 0$
5. $\mathsf{bits}(\mathsf{s}(0)) \to \mathsf{s}(0)$
6. $\mathsf{bits}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{bits}(\mathsf{s}(\mathsf{half}(x))))$

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4. $\mathsf{bits}(0) \to 0$
5. $\mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$

Negative coefficients

5. $\mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$
6. $\mathsf{bits}^{\#}(\mathsf{s}(x)) \to \mathsf{half}^{\#}(\mathsf{s}(x))$
7. $\mathsf{bits}^{\#}(\mathsf{s}(x)) \to \mathsf{bits}^{\#}(\mathsf{half}(\mathsf{s}(x)))$

Negative coefficients

5. $\mathsf{bits}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{bits}(\mathsf{half}(\mathsf{s}(x))))$
6. $\mathsf{bits}^{\#}(\mathsf{s}(x)) \to \mathsf{half}^{\#}(\mathsf{s}(x))$
7. $\mathsf{bits}^{\#}(\mathsf{s}(x)) \to \mathsf{bits}^{\#}(\mathsf{half}(\mathsf{s}(x)))$

No polynomial interpretation with non-negative coefficients

Negative coefficients

$s_{\mathcal{P}ol_3} = x_1 + 1$, $\text{bits}_{\mathcal{P}ol_3} = x_1 + 1$ and $\text{half}_{\mathcal{P}ol_3} = x_1 - 1$

5. $\text{bits}(\text{s}(\text{x})) \to \text{s}(\text{bits}(\text{half}(\text{s}(\text{x}))))$
6. $\text{bits}^{\#}(\text{s}(\text{x})) \to \text{half}^{\#}(\text{s}(\text{x}))$
7. $\text{bits}^{\#}(\text{s}(\text{x})) \to \text{bits}^{\#}(\text{half}(\text{s}(\text{x})))$

Negative coefficients

$\mathsf{s}_{\mathcal{P}ol_3} = x_1 + 1$, $\mathsf{bits}^{\#}_{\mathcal{P}ol_3} = x_1 + 1$ and $\mathsf{half}_{\mathcal{P}ol_3} = x_1 - 1$

6. $[\mathsf{bits}^{\#}(\mathsf{s}(\mathsf{x}))] = x + 1 > x = [\mathsf{half}^{\#}(\mathsf{s}(\mathsf{x}))]$
7. $[\mathsf{bits}^{\#}(\mathsf{s}(\mathsf{x}))] = x + 2 > x + 1 = [\mathsf{bits}^{\#}(\mathsf{half}(\mathsf{s}(\mathsf{x})))]$

Negative coefficients

- $[t_1] > [t_2] > [t_3] > \cdots > 0$

Negative coefficients

- $[t_1] > [t_2] > [t_3] > \cdots > 0$
- $[f(t_1, \ldots, t_n)] = \max(f_{\mathcal{P}ol}([t_1], \ldots, [t_n]), 0)$

Negative coefficients

- $[t_1] > [t_2] > [t_3] > \cdots > 0$
- $[f(t_1, \ldots, t_n)] = \max(f_{\mathcal{P}ol}([t_1], \ldots, [t_n]), 0)$
- $[u] > [v] \implies [u] - [v] > 0$

Negative coefficients

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- $[u] > [v] \implies [u] - [v] > 0$
- $[l]^{left} \leq [l] \leq [l]^{right}$

Negative coefficients

- $[t_1] > [t_2] > [t_3] > \cdots > 0$
- $[f(t_1, \ldots, t_n)] = \max(f_{\mathcal{P}ol}([t_1], \ldots, [t_n]), 0)$
- $[u] > [v] \implies [u] - [v] > 0$
- $[l]^{left} \leq [l] \leq [l]^{right}$
- for $l \to r$ we have $[l]^{left} \geq [r]^{right}$

Left and right

- $[\text{half}]_{\mathcal{P}ol}^{left} = x - 1$
- $[\text{half}]_{\mathcal{P}ol} = \max(x - 1, 0)$
- $[\text{half}]_{\mathcal{P}ol}^{right} = x$

Left and right

- $[\text{half}]^{left}_{\mathcal{P}ol} = x - 1$
- $[\text{half}]_{\mathcal{P}ol} = \max(x - 1, 0)$
- $[\text{half}]^{right}_{\mathcal{P}ol} = x$
- $x - 1 \leq \max(x - 1, 0) \leq x$

Left and right

- $[\text{half}]^{left}_{\mathcal{P}ol} = x - 1$
- $[\text{half}]_{\mathcal{P}ol} = \max(x - 1, 0)$
- $[\text{half}]^{right}_{\mathcal{P}ol} = x$
- $x - 1 \leq \max(x - 1, 0) \leq x$
- $[\text{half}]^{left}_{\mathcal{P}ol} \leq [\text{half}]_{\mathcal{P}ol} \leq [\text{half}]^{right}_{\mathcal{P}ol}$

Left and right

$$[t]^{left} = \begin{cases} t & \text{if } t \text{ is a variable} \\ 0 & \text{if } t = f(t_1, \ldots, t_n), ncon(p_1) = 0, \text{and } 0 > con(p_1) \\ p_1 & \text{if } t = f(t_1, \ldots, t_n), \text{otherwise} \end{cases}$$

Where $p_1 = f_{\mathcal{P}ol}([t_1]^{left}, \ldots, [t_n]^{left})$

Left and right

$$[t]^{right} = \begin{cases} t & \text{if } t \text{ is a variable} \\ ncon(p_2) & \text{if } t = f(t_1, \ldots, t_n), \text{and } 0 > con(p_2) \\ p_2 & \text{if } t = f(t_1, \ldots, t_n), \text{otherwise} \end{cases}$$

Where $p_2 = f_{\mathcal{P}ol}([t_1]^{right}, \ldots [t_n]^{right})$

Left and right

$s_{\mathcal{P}ol_3} = x_1 + 1$, $bits_{\mathcal{P}ol_3} = x_1 + 1$ and $half_{\mathcal{P}ol_3} = max(x_1 - 1, 0)$

- $\text{bits}^\#(\text{s}(\text{x})) \to \text{half}^\#(\text{s}(\text{x}))$
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- $[\text{bits}^{\#}(\text{s}(\text{x}))]^{left} = x + 2 > x = [\text{half}^{\#}(\text{s}(\text{x}))]^{right}$
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- $[bits^{\#}(s(x))]^{left} = x + 2 > x = [half^{\#}(s(x))]^{right}$
- $[bits^{\#}(s(x))]^{left} = x + 2 > x + 1 = [bits^{\#}(half(s(x)))]^{right}$
- Since $con(half^{\#}_{\mathcal{P}ol}([s(x)])) = con(half^{\#}_{\mathcal{P}ol}(x + 1)) \geq 0$

Left and right

- Only for concrete polynomials

Left and right

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- $\text{half}_{\mathcal{P}ol} = ax + b$
- $ncon(p_i) = ax =^? 0$ and $con(p_i) = b <^? 0$

Left and right

$$[t]^{left} = \begin{cases} t & \text{if } t \text{ is a variable} \\ 0 & \text{if } t = f(t_1, \ldots, t_n), ncon(p_1) = 0, \text{and } 0 > con(p_1) \\ p_1 & \text{if } t = f(t_1, \ldots, t_n), \text{otherwise} \end{cases}$$

Where $p_1 = f_{\mathcal{P}ol}([t_1]^{left}, \ldots, [t_n]^{left})$

$$[t]^{right} = \begin{cases} t & \text{if } t \text{ is a variable} \\ ncon(p_2) & \text{if } t = f(t_1, \ldots, t_n), \text{and } 0 > con(p_2) \\ p_2 & \text{if } t = f(t_1, \ldots, t_n), \text{otherwise} \end{cases}$$

Where $p_2 = f_{\mathcal{P}ol}([t_1]^{right}, \ldots [t_n]^{right})$

If t is a variable then $[t]^{left} = t = [t]^{right}$ and $\alpha_t^{left} = true = \alpha_t^{right}$

Left and right

$$[t]^{left} = \begin{cases} t & \text{if } t \text{ is a variable} \\ 0 & \text{if } t = f(t_1, \ldots, t_n), ncon(p_1) = 0, \text{and } 0 > con(p_1) \\ p_1 & \text{if } t = f(t_1, \ldots, t_n), \text{otherwise} \end{cases}$$

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If $t = f(t_1, \ldots t_n)$, then $[t]^{left} = ncon(p_1) + \boldsymbol{b_t^{left}}$,
$\alpha_t^{left} = \alpha_{t_1}^{left} \wedge \ldots \alpha_{t_n}^{left} \wedge (\alpha_{ncon(p_1)=0} \wedge 0 > con(p_1) \rightarrow \boldsymbol{b_t^{left}} = 0)$
$\wedge(\neg(\alpha_{ncon(p_1)=0} \wedge 0 > con(p_1)) \rightarrow \boldsymbol{b_t^{left}} = con(p_1))$

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Where $p_2 = f_{\mathcal{P}ol}([t_1]^{right}, \ldots [t_n]^{right})$

If $t = f(t_1, \ldots t_n)$, then $[t]^{right} = ncon(p_2) + b_t^{right}$,
$\alpha_t^{right} = \alpha_{t_1}^{right} \wedge \ldots \alpha_{t_n}^{right} \wedge (0 > con(p_2) \rightarrow b_t^{right} = 0)$
$\wedge(\neg(0 > con(p_2)) \rightarrow b_t^{right} = con(p_2))$

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- Fix n and remove **bold** variables $\boldsymbol{a}$ by "$a - n$" for a fresh a. $\{-n, \dots, 2^k - 1 - n\} \implies \{0, \dots, 2^k - 1\}$

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- Remove $\geq$ and subtractions from the constraint and check the satisfiability using SAT

Results

			APProVE-SAT			AProVE-PB			AProVE 1.2		
Limit	Range	Degree	Yes	TO	Time	Yes	TO	Time	Yes	TO	Time
60s	1	1	421	0	45.5	421	0	61.6	421	1	151.8
60s	2	1	431	0	91.8	431	0	158.5	414	48	3633.2
60s	3	1	434	0	118.6	434	1	222.1	408	81	5793.2
60s	3	sm	440	51	5585.9	427	82	7280.3	404	171	11608.1
10m	1	1	421	0	45.5	421	0	61.6	421	1	691.8
10m	2	1	431	0	91.8	431	0	158.5	418	41	27888.4
10m	3	1	434	0	118.6	434	0	689.6	415	53	38286.4

			AProVE-CLP			AProVE-CiME			TTT		
Limit	Range	Degree	Yes	TO	Time	Yes	TO	Time	Yes	TO	Time
60s	1	1	420	16	1357.8	408	1	168.3	326	32	2568.5
60s	2	1	420	37	3558.3	408	43	3201.0	335	83	5677.6
60s	3	1	407	91	6459.5	402	67	5324.1	338	110	7426.9
60s	3	sm	367	145	10357.4	361	147	10107.7			
10m	1	1	421	11	7852.2	408	0	332.7	328	16	14007.8
10m	2	1	423	25	18795.6	412	33	22190.4	337	68	45046.6
10m	3	1	420	51	41493.8	407	46	33873.6	340	91	61209.2

Questions