Representing 'undefined' in lambda calculus

Marten Straatsma
s1041007

Institute for Computing and Information Sciences
Mathematical Foundation of Computer Science
Radboud University Nijmegen

Presentation MFoCS Seminar
23 January 2023

Radboud University Nijmegen

J. Functional Programming **2** (3): 367–374, July 1992 © 1992 Cambridge University Press

THEORETICAL PEARLS

Representing 'undefined' in lambda calculus

HENK BARENDREGT
Faculty of Mathematics and Computer Science, Catholic University Nijmegen, Toernooiveld 1, 6525 ED, The Netherlands
(e-mail: *henk*@cs.kun.nl)

Abstract

Let ψ be a partial recursive function (of one argument) with λ-defining term $F \in \Lambda^\circ$. This means

$$\psi(n) = m \Leftrightarrow F\ulcorner n \urcorner =_\beta \ulcorner m \urcorner.$$

There are several proposals for what $F\ulcorner n \urcorner$ should be in case $\psi(n)$ is undefined: (1) a term without a normal form (Church); (2) an unsolvable term (Barendregt); (3) an easy term (Visser); (4) a term of order 0 (Statman).

These four possibilities will be covered by one 'master' result of Statman which is based on the 'Anti Diagonal Normalization Theorem' of Visser (1980). That ingenious theorem about

J. Functional Programming **2** (3): 367–374, July 1992 © 1992 Cambridge University Press

THEORETICAL PEARLS

Representing 'undefined' in lambda calculus

NUMERATIONS, λ-CALCULUS & ARITHMETIC

Albert Visser

Mathematical Institute, Budapestlaan 6
3508 TA Utrecht, The Netherlands.

Dedicated to H.B. Curry on the occasion of his 80th Birthday

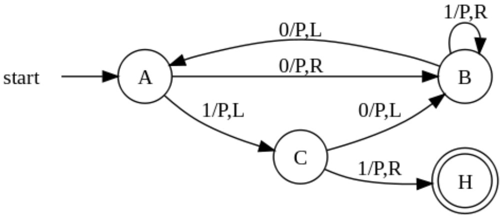
ABSTRACT

Applications of complete and precomplete numerations (as

The Problem
Models of Computability

The Problem
Models of Computability

1. Partial Computable Functions



The Problem
Models of Computability

1. Partial Computable Functions
2. General Recursive Functions

$$0, S, \pi_n^i$$

$$f = g \circ h$$

$$f(0, \vec{x}) = g(\vec{x}), f(n+1, \vec{x}) = h(f(n, \vec{x}), \vec{x}, n)$$

$$f = \mu n.R(n, \vec{x})$$

The Problem
Models of Computability

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$M ::= x \mid \lambda x.M \mid MM$$

$$\lambda x.M[x] \to_\alpha \lambda y.M[y]$$

$$(\lambda x.M[x])N \to_\beta M[x := N]$$

The Problem
Models of Computability

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$M ::= x \mid \lambda x.M \mid MM$$

$$\lambda x.M[x] \to_\alpha \lambda y.M[y]$$

$$(\lambda x.M[x])N \to_\beta M[x := N]$$

$$\underline{n} = \lambda f.\ \lambda x.\ f^n x$$

The Problem

‘undefined’ lambda terms?

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

The Problem
'undefined' lambda terms?

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$\forall f \in \mathcal{R} \;\; \exists F \in \Lambda^{\circ} \;\; \forall n \in \mathbb{N} \;\; F\underline{n} =_{\beta} \underline{f(n)}$$

The Problem
'undefined' lambda terms?

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$\forall f \in \mathcal{R} \;\; \exists F \in \Lambda^{\circ} \;\; \forall n \in \mathbb{N} \;\; F\underline{n} =_{\beta} \underline{f(n)}$$

$$\forall F \in \Lambda^{\circ}, n \in \mathbb{N} \;\; \exists M \in \Lambda^{\circ} \;\; F\underline{n} =_{\beta} M$$

The Problem

'undefined' lambda terms?

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$\forall f \in \mathcal{R} \;\; \exists F \in \Lambda^\circ \;\; \forall n \in \mathbb{N} \;\; F\underline{n} =_\beta \underline{f(n)}$$

$$\forall F \in \Lambda^\circ, n \in \mathbb{N} \;\; \exists M \in \Lambda^\circ \;\; F\underline{n} =_\beta M$$

$$\varphi(n)\uparrow ?$$

The Problem
'undefined' lambda terms?

1. Partial Computable Functions
2. General Recursive Functions
3. Closed Lambda Terms

$$\forall f \in \mathcal{R} \;\; \exists F \in \Lambda^\circ \;\; \forall n \in \mathbb{N} \;\; F\underline{n} =_\beta \underline{f(n)}$$

$$\forall F \in \Lambda^\circ, n \in \mathbb{N} \;\; \exists M \in \Lambda^\circ \;\; F\underline{n} =_\beta M$$

$$\varphi(n)\uparrow ?$$

$$\omega = \lambda x.xx$$

$$\omega\omega = (\lambda x.xx)\omega \to_\beta (xx)[x := \omega] = \omega\omega$$

Outline

Notation

Numerations
- Numerations
- Precomplete Numerations
- The Fixed-Point Theorem

The Anti Diagonal Normalisation Theorem
- Ordering Predicates
- Anti Diagonal Functions
- The Anti Diagonal Normalisation Theorem

Notions of ‘undefined’
- à la Visser
- à la Barendregt

Notation

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$
2. $\mathcal{PR}$ unary partial recursive functions
   $\psi(n)\downarrow$ denotes $\psi(n)$ is defined
   $\psi(n)\uparrow$ denotes $\psi(n)$ is undefined

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$
2. $\mathcal{PR}$ unary partial recursive functions
   $\psi(n) \downarrow$ denotes $\psi(n)$ is defined
   $\psi(n) \uparrow$ denotes $\psi(n)$ is undefined
3. $\mathcal{R}$ unary total recursive functions

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$
2. $\mathcal{PR}$ unary partial recursive functions
   $\psi(n) \downarrow$ denotes $\psi(n)$ is defined
   $\psi(n) \uparrow$ denotes $\psi(n)$ is undefined
3. $\mathcal{R}$ unary total recursive functions
4. Λ set of λ-terms
   Λ° set of closed λ-terms

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$
2. $\mathcal{PR}$ unary partial recursive functions
   $\psi(n) \downarrow$ denotes $\psi(n)$ is defined
   $\psi(n) \uparrow$ denotes $\psi(n)$ is undefined
3. $\mathcal{R}$ unary total recursive functions
4. Λ set of λ-terms
   Λ° set of closed λ-terms
5. $\underline{n}$: from codes to objects
   $\ulcorner N \urcorner$: from objects to codes

Notation

1. $\mathbb{N} = \{0, 1, 2, \ldots\}$
2. $\mathcal{PR}$ unary partial recursive functions
   $\psi(n) \downarrow$ denotes $\psi(n)$ is defined
   $\psi(n) \uparrow$ denotes $\psi(n)$ is undefined
3. $\mathcal{R}$ unary total recursive functions
4. Λ set of λ-terms
   Λ° set of closed λ-terms
5. $\underline{n}$: from codes to objects
   $\ulcorner N \urcorner$: from objects to codes
6. $\boldsymbol{E} \in \Lambda^\circ$ self interpreter
   $\boldsymbol{E}\ \underline{\ulcorner M \urcorner} =_\beta M$

Outline

Notation

Numerations
- Numerations
- Precomplete Numerations
- The Fixed-Point Theorem

The Anti Diagonal Normalisation Theorem
- Ordering Predicates
- Anti Diagonal Functions
- The Anti Diagonal Normalisation Theorem

Notions of 'undefined'
- à la Visser
- à la Barendregt

Numerations
Definitions

Definitions numeration

1. A *numeration* is a pair $\gamma = (\nu, S)$ with $\nu : \mathbb{N} \twoheadrightarrow S$

Numerations

Definitions

Definitions numeration

1. A *numeration* is a pair $\gamma = (\nu, S)$ with $\nu : \mathbb{N} \twoheadrightarrow S$
2. $n \sim_\gamma m \iff \nu(n) = \nu(m)$

Numerations
Definitions

Definitions numeration

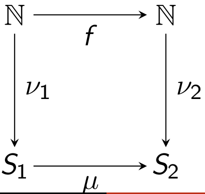
1. A *numeration* is a pair $\gamma = (\nu, S)$ with $\nu : \mathbb{N} \twoheadrightarrow S$
2. $n \sim_\gamma m \iff \nu(n) = \nu(m)$
3. $\gamma_1 = (\nu_1, S_1)$, $\gamma_2 = (\nu_2, S_2)$
   A map $\mu : S_1 \to S_2$ is a *morphism* from γ_1 to γ_2 if for some $f \in \mathcal{R}$:
   $$\nu_2 \circ f = \mu \circ \nu_2$$
   Notation: $\mu : \gamma_1 \to \gamma_2$

Numerations

Definitions

Definitions numeration

1. A *numeration* is a pair $\gamma = (\nu, S)$ with $\nu : \mathbb{N} \twoheadrightarrow S$
2. $n \sim_\gamma m \iff \nu(n) = \nu(m)$
3. $\gamma_1 = (\nu_1, S_1)$, $\gamma_2 = (\nu_2, S_2)$
   A map $\mu : S_1 \to S_2$ is a *morphism* from γ_1 to γ_2 if for some $f \in \mathcal{R}$:
   $$\nu_2 \circ f = \mu \circ \nu_2$$
   Notation: $\mu : \gamma_1 \to \gamma_2$



Numerations
Examples

Examples of numerations

Numerations
Examples

Examples of numerations

1. $\boldsymbol{\Lambda_\beta} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\,\underline{n}$

Numerations
Examples

Examples of numerations

1. $\boldsymbol{\Lambda_\beta} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\,\underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_n$

Numerations
Examples

Examples of numerations

1. $\boldsymbol{\Lambda}_{\boldsymbol{\beta}} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\,\underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_n$
3. $T \supseteq PA,\ A(\vec{x}) \in \Sigma^0_n(\vec{x})$

Numerations

Examples

Examples of numerations

1. $\boldsymbol{\Lambda}_{\boldsymbol{\beta}} = (E, \Lambda^{\circ}/ =_{\beta})$
   $E(n) = \boldsymbol{E}\,\underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_{\boldsymbol{n}}$
3. $T \supseteq PA$, $A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$[\![A(\vec{x})]\!]^T_{n,\vec{x}} = \{B(\vec{x}) \in \Sigma^0_n(\vec{x}) \mid T \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow B(\vec{x})\}$$

Numerations

Examples

Examples of numerations

1. $\mathbf{\Lambda}_{\boldsymbol{\beta}} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\,\underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_n$
3. $T \supseteq PA,\ A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$[\![A(\vec{x})]\!]^T_{n,\vec{x}} = \{B(\vec{x}) \in \Sigma^0_n(\vec{x}) \mid T \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow B(\vec{x})\}$$

$$sig^T_{n,\vec{x}}(\ulcorner A(\vec{x}) \urcorner) = [\![A(\vec{x})]\!]^T_{n,\vec{x}}$$

Numerations
Examples

Examples of numerations

1. $\mathbf{\Lambda}_{\boldsymbol{\beta}} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\, \underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_n$
3. $T \supseteq PA,\ A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$[\![A(\vec{x})]\!]^T_{n,\vec{x}} = \{B(\vec{x}) \in \Sigma^0_n(\vec{x}) \mid T \vdash \forall \vec{x}\ \ A(\vec{x}) \leftrightarrow B(\vec{x})\}$$
$$sig^T_{n,\vec{x}}(\ulcorner A(\vec{x}) \urcorner) = [\![A(\vec{x})]\!]^T_{n,\vec{x}}$$
$$Sig^T_{n,\vec{x}} = \{[\![A(\vec{x})]\!]^T_{n,\vec{x}} \mid A(\vec{x}) \in \Sigma^0_n(\vec{x})\}$$

Numerations
Examples

Examples of numerations

1. $\boldsymbol{\Lambda_\beta} = (E, \Lambda^\circ / =_\beta)$
   $E(n) = \boldsymbol{E}\,\underline{n}$
2. $\boldsymbol{PR} = (\Phi, \mathcal{PR})$
   $\Phi(n) = \varphi_n$
3. $T \supseteq PA,\ A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$[\![A(\vec{x})]\!]^T_{n,\vec{x}} = \{B(\vec{x}) \in \Sigma^0_n(\vec{x}) \mid T \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow B(\vec{x})\}$$
$$sig^T_{n,\vec{x}}(\ulcorner A(\vec{x}) \urcorner) = [\![A(\vec{x})]\!]^T_{n,\vec{x}}$$
$$Sig^T_{n,\vec{x}} = \{[\![A(\vec{x})]\!]^T_{n,\vec{x}} \mid A(\vec{x}) \in \Sigma^0_n(\vec{x})\}$$
$$\mathcal{S}ig^T_{n,\vec{x}} = (sig^T_{n,\vec{x}}, Sig^T_{n,\vec{x}})$$

Precomplete Numerations
Definitions

Properties of numerations

Precomplete Numerations
Definitions

Properties of numerations

1. γ is *positive* if $\sim_\gamma$ is recursively enumerable

Precomplete Numerations

Definitions

Properties of numerations

1. γ is *positive* if $\sim_\gamma$ is recursively enumerable
2. γ is *precomplete* if

$$\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; \forall n \in \mathbb{N} \;\; \psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n)$$

Precomplete Numerations
Definitions

Properties of numerations

1. γ is *positive* if $\sim_\gamma$ is recursively enumerable
2. γ is *precomplete* if

$$\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; \forall n \in \mathbb{N} \;\; \psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n)$$

3. γ is *complete* if

$$\exists a \in S \;\; \forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; \forall n \in \mathbb{N} \;\; \begin{array}{l} \psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n) \\ \psi(n)\uparrow \implies \nu(\psi(n)) = a \end{array}$$

Precomplete Numerations
Examples

Proposition

1. $\boldsymbol{\Lambda}_\beta$ is precomplete
2. $\boldsymbol{PR}$ is precomplete
3. $\mathcal{S}ig^T_{n,\vec{x}}$ is precomplete

Precomplete Numerations
Examples

$\mathbf{\Lambda}_{\beta}$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^{\circ}$ be a λ-defining term for ψ.

□

Precomplete Numerations
Examples

$\boldsymbol{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$.

□

Precomplete Numerations

Examples

$\mathbf{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$. Then if $\psi(n)\downarrow$:

$$\boldsymbol{E}\underline{f(n)} =_\beta \boldsymbol{E}\underline{\ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner}$$

□

Precomplete Numerations

Examples

$\mathbf{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$. Then if $\psi(n)\downarrow$:

$$\boldsymbol{E}\underline{f(n)} =_\beta \boldsymbol{E}\underline{\ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner}$$
$$=_\beta \boldsymbol{E} \circ F\underline{n}$$

□

Precomplete Numerations
Examples

$\mathbf{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$. Then if $\psi(n)\downarrow$:

$$\begin{aligned} \boldsymbol{E}\underline{f(n)} &=_\beta \boldsymbol{E}\underline{\ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner} \\ &=_\beta \boldsymbol{E} \circ F\underline{n} \\ &=_\beta \boldsymbol{E}(F\underline{n}) \end{aligned}$$

□

Precomplete Numerations

Examples

$\boldsymbol{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$. Then if $\psi(n)\downarrow$:

$$\begin{aligned}
\boldsymbol{E}\underline{f(n)} &=_\beta \boldsymbol{E}\underline{\ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner} \\
&=_\beta \boldsymbol{E} \circ F\underline{n} \\
&=_\beta \boldsymbol{E}(F\underline{n}) \\
&=_\beta \boldsymbol{E}\underline{\psi(n)}
\end{aligned}$$

□

Precomplete Numerations

Examples

$\boldsymbol{\Lambda}_\beta$ is precomplete

Proof.

1. Given $\psi \in \mathcal{PR}$ let $F \in \Lambda^\circ$ be a λ-defining term for ψ. Define $f(n) = \ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner$. Then if $\psi(n)\downarrow$:

$$\begin{aligned}\boldsymbol{E}\underline{f(n)} &=_\beta \boldsymbol{E}\underline{\ulcorner \boldsymbol{E} \circ F\underline{n} \urcorner} \\ &=_\beta \boldsymbol{E} \circ F\underline{n} \\ &=_\beta \boldsymbol{E}(F\underline{n}) \\ &=_\beta \boldsymbol{E}\underline{\psi(n)}\end{aligned}$$

Hence $f(n) \sim_\gamma \psi(n)$.

□

Precomplete Numerations
Examples

PR is precomplete

Proof.

2. Given $\psi \in \mathcal{PR}$ define

$$\theta(n, m) = \varphi_{\psi(n)}(m)$$

Precomplete Numerations
Examples

PR is precomplete

Proof.

2. Given $\psi \in \mathcal{PR}$ define

$$\theta(n, m) = \varphi_{\psi(n)}(m)$$

By the S-m-n theorem there exists an $f \in \mathcal{R}$ such that

$$\theta(n, m) = \varphi_{f(n)}(m)$$

Precomplete Numerations

Examples

PR is precomplete

Proof.

2. Given $\psi \in \mathcal{PR}$ define

$$\theta(n, m) = \varphi_{\psi(n)}(m)$$

By the S-m-n theorem there exists an $f \in \mathcal{R}$ such that

$$\theta(n, m) = \varphi_{f(n)}(m)$$

Then

$$\begin{aligned} \psi(n)\downarrow &\implies \varphi_{f(n)} = \varphi_{\psi(n)} \\ &\implies f(n) \sim_\gamma \psi(n) \end{aligned}$$

Precomplete Numerations

Examples

$\mathcal{S}ig^0_{n,\vec{x}}$ is precomplete

Proof.

3. There is a $T_n(y, \vec{x}) \in \Sigma^0_n(y, \vec{x})$ s.t. for every $A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$PA \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow T_n(\ulcorner A(\vec{x}) \urcorner, \vec{x})$$

□

Precomplete Numerations
Examples

$\mathcal{S}ig^0_{n,\vec{x}}$ is precomplete

Proof.

3. There is a $T_n(y,\vec{x}) \in \Sigma^0_n(y,\vec{x})$ s.t. for every $A(\vec{x}) \in \Sigma^0_n(\vec{x})$
$$PA \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow T_n(\ulcorner A(\vec{x}) \urcorner, \vec{x})$$

Consider $\psi \in \mathcal{PR}$.
Define $f(m) = \ulcorner \exists y \;\; \psi(m) = y \;\&\; T_n(y,\vec{x}) \urcorner$.

□

Precomplete Numerations

Examples

$\mathcal{S}ig^0_{n,\vec{x}}$ is precomplete

Proof.

3. There is a $T_n(y,\vec{x}) \in \Sigma^0_n(y,\vec{x})$ s.t. for every $A(\vec{x}) \in \Sigma^0_n(\vec{x})$

$$PA \vdash \forall \vec{x} \;\; A(\vec{x}) \leftrightarrow T_n(\ulcorner A(\vec{x}) \urcorner, \vec{x})$$

Consider $\psi \in \mathcal{PR}$.
Define $f(m) = \ulcorner \exists y \;\; \psi(m) = y \;\&\; T_n(y,\vec{x}) \urcorner$.
If $\psi(m)\downarrow = \ulcorner A(\vec{x}) \urcorner$, then

$$T \vdash \forall \vec{x} (\exists y \;\; \psi(m) = y \;\&\; T_n(y,\vec{x}) \leftrightarrow T_n(\ulcorner A(\vec{x}) \urcorner, \vec{x}) \leftrightarrow A(\vec{x}))$$

□

Fixed-point Theorem
Theorem

Fixed-point theorem

If γ is precomplete, then

$$\forall f \in \mathcal{R} \;\; \exists n \in \mathbb{N} \;\; f(n) \sim_\gamma n$$

Fixed-point Theorem
Theorem

Fixed-point theorem

If γ is precomplete, then

$$\forall f \in \mathcal{R} \;\; \exists n \in \mathbb{N} \;\; f(n) \sim_\gamma n$$

Proof.

Given $f \in \mathcal{R}$ define $\psi(m) = f(\varphi_m(m))$.

□

Fixed-point Theorem
Theorem

Fixed-point theorem

If γ is precomplete, then

$$\forall f \in \mathcal{R} \ \ \exists n \in \mathbb{N} \ \ f(n) \sim_\gamma n$$

Proof.

Given $f \in \mathcal{R}$ define $\psi(m) = f(\varphi_m(m))$.
Choose $h = \varphi_e \in \mathcal{R}$ totalising ψ modulo $\sim_\gamma$.

□

Fixed-point Theorem

Theorem

Fixed-point theorem

If γ is precomplete, then

$$\forall f \in \mathcal{R} \;\; \exists n \in \mathbb{N} \;\; f(n) \sim_\gamma n$$

Proof.

Given $f \in \mathcal{R}$ define $\psi(m) = f(\varphi_m(m))$.
Choose $h = \varphi_e \in \mathcal{R}$ totalising ψ modulo $\sim_\gamma$.
Define $n = h(e)$.

□

Fixed-point Theorem

Theorem

Fixed-point theorem

If γ is precomplete, then

$$\forall f \in \mathcal{R} \;\; \exists n \in \mathbb{N} \;\; f(n) \sim_\gamma n$$

Proof.

Given $f \in \mathcal{R}$ define $\psi(m) = f(\varphi_m(m))$.
Choose $h = \varphi_e \in \mathcal{R}$ totalising ψ modulo $\sim_\gamma$.
Define $n = h(e)$. Then also $\psi(e)\downarrow$ and

$$n = h(e) \sim_\gamma \psi(e) = f(\varphi_e(e)) = f(n)$$

□

Fixed-point Theorem
Corollaries

Corollary

Fixed-point Theorem

Corollaries

Corollary

1. Fixed-point theorem in λ-calculus

$$\forall F \in \Lambda^\circ \quad \exists N \in \Lambda^\circ \quad FN =_\beta N$$

Fixed-point Theorem
Corollaries

Corollary

1. Fixed-point theorem in λ-calculus

$$\forall F \in \Lambda^\circ \;\; \exists N \in \Lambda^\circ \;\; FN =_\beta N$$

2. Recursion theorem

$$\forall f \in \mathcal{R} \;\; \exists n \in \mathbb{N} \;\; \varphi_{f(n)} = \varphi_n$$

Fixed-point Theorem
Corollaries

Corollary

1. Fixed-point theorem in λ-calculus

$$\forall F \in \Lambda^\circ \ \ \exists N \in \Lambda^\circ \ \ FN =_\beta N$$

2. Recursion theorem

$$\forall f \in \mathcal{R} \ \ \exists n \in \mathbb{N} \ \ \varphi_{f(n)} = \varphi_n$$

3. $\ulcorner x \urcorner \sim_T \ulcorner y \urcorner \iff x, y$ provably equivalent in $T \supset PA$
   Corresponding γ_T not precomplete

Fixed-point Theorem

Corollaries

Proof.

1. Apply fixed-point theorem to $\boldsymbol{\Lambda}_\beta$.
   Use $f(n) = \underline{F(\boldsymbol{E}\ulcorner \underline{n} \urcorner)}$.
2. Apply fixed-point theorem to $\boldsymbol{PR}$.

□

Fixed-point Theorem
Corollaries

Proof.

1. Apply fixed-point theorem to $\boldsymbol{\Lambda}_\beta$.
   Use $f(n) = \underline{F(\boldsymbol{E}\ulcorner \underline{n} \urcorner)}$.
2. Apply fixed-point theorem to $\boldsymbol{PR}$.
3. Suppose γ_T precomplete. Define $f(\ulcorner \varphi \urcorner) = \ulcorner \neg\varphi \urcorner$.
   By the fixed-point-theorem there exists a $\ulcorner \varphi \urcorner$ such that

$$\ulcorner \varphi \urcorner \sim_T f(\ulcorner \varphi \urcorner) = \ulcorner \neg\varphi \urcorner$$

□

Outline

Notation

Numerations
- Numerations
- Precomplete Numerations
- The Fixed-Point Theorem

The Anti Diagonal Normalisation Theorem
- Ordering Predicates
- Anti Diagonal Functions
- The Anti Diagonal Normalisation Theorem

Notions of ‘undefined’
- à la Visser
- à la Barendregt

Ordering Predicates

Definition

Let $Q_1(n) \iff \exists m\ R_1(n,m)$ and $Q_2(n) \iff \exists m\ R_2(n,m)$ be r.e. predicates. Write

$$Q_1(n_1) \leqslant Q_2(n_2) \iff \exists m\ R_1(n_1,m)\ \&\ \forall m' < m\ \neg R_2(n_2,m')$$

$$Q_1(n_1) < Q_2(n_2) \iff \exists m\ R_1(n_1,m)\ \&\ \forall m' \leqslant m\ \neg R_2(n_2,m')$$

Informally:

$$Q_1(n_1) \leqslant Q_2(n_2) \iff \mu m.R_1(n_1,m) \leqslant \mu m.R_2(n_2,m)$$

$$Q_1(n_1) < Q_2(n_2) \iff \mu m.R_1(n_1,m) < \mu m.R_2(n_2,m)$$

Ordering Predicates

Lemma

1. $Q_1(n_1) \leqslant Q_2(n_2)$ and $Q_1(n_1) < Q_2(n_2)$ are r.e. relations.
2. $Q_1(n_1) \vee Q_2(n_2) \implies Q_1(n_1) \leqslant Q_2(n_2) \vee Q_2(n_2) < Q_1(n_1)$

Anti Diagonal Functions

Definition (anti) diagonal function

An *(anti) diagonal function* (w.r.t. γ) is a $\delta \in \mathcal{PR}$ such that for all $n \in \mathbb{N}$

$$\delta(n)\downarrow \implies \delta(n) \not\sim_\gamma n$$

The Anti Diagonal Normalisation Theorem

Anti Diagonal Normalisation Theorem

Let γ be a precomplete numeration and $\delta \in \mathcal{PR}$ an anti diagonal function. Then

$$\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; \forall n \in \mathbb{N} \; \begin{array}{l} \psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n) \\ \psi(n)\uparrow \implies f(n) \notin \mathrm{dom}(\delta) \end{array}$$

We say that f *totalises* ψ *modulo* $\sim_\gamma$ *avoiding* $\mathrm{dom}(\delta)$.

The Anti Diagonal Normalisation Theorem

Proof.

The Anti Diagonal Normalisation Theorem

Proof.

Given $\psi \in \mathcal{PR}$, define $\theta(n) = \varphi_n(n)$ also $\mathcal{PR}$.

The Anti Diagonal Normalisation Theorem

Proof.

Given $\psi \in \mathcal{PR}$, define $\theta(n) = \varphi_n(n)$ also $\mathcal{PR}$.
γ precomplete $\implies \exists g \in \mathcal{R}\ \forall n \in \mathbb{N}\ \ \varphi_n(n)\downarrow \implies g(n) \sim_\gamma \varphi_n(n)$

The Anti Diagonal Normalisation Theorem

Proof.

Given $\psi \in \mathcal{PR}$, define $\theta(n) = \varphi_n(n)$ also $\mathcal{PR}$.
γ precomplete $\implies \exists g \in \mathcal{R}\ \forall n \in \mathbb{N}\ \ \varphi_n(n)\downarrow \implies g(n) \sim_\gamma \varphi_n(n)$
By the S-m-n theorem there exists an $s \in \mathcal{R}$ such that

$$\varphi_{s(n)}(m) = \begin{cases} \psi(n) & \text{if } \psi(n)\downarrow \leqslant \delta(g(m))\downarrow \\ \delta(g(m)) & \text{if } \delta(g(m))\downarrow < \psi(m)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

$\square$

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \not\sim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\psi(n)\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$$

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\begin{aligned} \psi(n)\downarrow &\implies \varphi_{s(n)}(s(n)) = \psi(n) \\ &\implies g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \psi(n) \end{aligned}$$

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\begin{aligned} \psi(n)\downarrow &\implies \varphi_{s(n)}(s(n)) = \psi(n) \\ &\implies g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \psi(n) \end{aligned}$$

$$\psi(n)\uparrow \implies \varphi_{s(n)}(s(n))\uparrow$$

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\begin{aligned} \psi(n)\downarrow &\implies \varphi_{s(n)}(s(n)) = \psi(n) \\ &\implies g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \psi(n) \end{aligned}$$

$$\begin{aligned} \psi(n)\uparrow &\implies \varphi_{s(n)}(s(n))\uparrow \\ &\implies \delta(g(s(n)))\uparrow \end{aligned}$$

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\begin{aligned}\psi(n)\downarrow &\implies \varphi_{s(n)}(s(n)) = \psi(n)\\ &\implies g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \psi(n)\end{aligned}$$

$$\begin{aligned}\psi(n)\uparrow &\implies \varphi_{s(n)}(s(n))\uparrow\\ &\implies \delta(g(s(n)))\uparrow\\ &\implies g(s(n)) \notin \mathrm{dom}(\delta)\end{aligned}$$

□

Proof.

Suppose that $\varphi_{s(n)}(s(n))\downarrow$ and $\varphi_{s(n)}(s(n)) = \delta(g(s(n)))$. Then

$$g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \delta(g(s(n))) \nsim_\gamma g(s(n))$$

Therefore $\varphi_{s(n)}(s(n))\downarrow \implies \varphi_{s(n)}(s(n)) = \psi(n)$.

$$\begin{aligned}\psi(n)\downarrow &\implies \varphi_{s(n)}(s(n)) = \psi(n)\\ &\implies g(s(n)) \sim_\gamma \varphi_{s(n)}(s(n)) = \psi(n)\end{aligned}$$

$$\begin{aligned}\psi(n)\uparrow &\implies \varphi_{s(n)}(s(n))\uparrow\\ &\implies \delta(g(s(n)))\uparrow\\ &\implies g(s(n)) \notin \mathrm{dom}(\delta)\end{aligned}$$

Therefore $f = g \circ s$ totalises ψ modulo $\sim_\gamma$ avoiding $\mathrm{dom}(\delta)$. □

Outline

Notation

Numerations
- Numerations
- Precomplete Numerations
- The Fixed-Point Theorem

The Anti Diagonal Normalisation Theorem
- Ordering Predicates
- Anti Diagonal Functions
- The Anti Diagonal Normalisation Theorem

Notions of 'undefined'
- à la Visser
- à la Barendregt

'undefined' à la Visser

Definition of ε

Let $R(x, \vec{y}) = \exists z\ R_0(z, x, \vec{y})$ be an r.e. relation.
Define $\varepsilon x.\ R(x, \vec{y}) = (\mu u.\ R_0(u_0, u_1, \vec{y}))_1$.
Informally: $\varepsilon x.\ R(x, \vec{y})$ gives an element of $\{x \mid R(x, \vec{y})\}$ if there is one.

'undefined' à la Visser

Visser's Theorem (simplified)

Let $U \subseteq \mathbb{N}$ be a set of (codes of) identities between λ-terms such that

$$\ulcorner M = N \urcorner \in U \quad \lambda\beta \nvdash M = N$$

Let
$I = \{\Omega_0 \in \Lambda \mid \forall P \in \Lambda, \ulcorner M = N \urcorner \in U \quad \lambda\beta + (\Omega_0 = P) \nvdash M = N\}$.

Then there is an $F \in \Lambda^\circ$ such that for all $p, m, n \in \mathbb{N}$:

$$\varphi_p(m) = n \iff \lambda\beta \vdash F \ \underline{p} \ \underline{m} = \underline{n}$$

$$\varphi_p(m) \uparrow \iff \exists \Omega_0 \in I \quad \lambda\beta \vdash F \ \underline{p} \ \underline{m} = \Omega_0$$

'undefined' à la Visser

Proof.

$$\delta(\ulcorner P \urcorner) = \varepsilon \ulcorner Q \urcorner . \ \ (\exists \ulcorner M = N \urcorner \in U \ \ \lambda\beta + (P = Q) \vdash M = N)$$

'undefined' à la Visser

Proof.

$$\delta(\ulcorner P \urcorner) = \varepsilon \ulcorner Q \urcorner. \ \ (\exists \ulcorner M = N \urcorner \in U \ \ \lambda\beta + (P = Q) \vdash M = N)$$

δ is an anti diagonal function for $\mathbf{\Lambda}_{\beta}$ & $\ulcorner P \urcorner \notin \mathrm{dom}(\delta) \iff P \in I$

'undefined' à la Visser

Proof.

$$\delta(\ulcorner P \urcorner) = \varepsilon \ulcorner Q \urcorner. \ \ (\exists \ulcorner M = N \urcorner \in U \ \ \lambda\beta + (P = Q) \vdash M = N)$$

δ is an anti diagonal function for $\mathbf{\Lambda}_\beta$ & $\ulcorner P \urcorner \notin \mathrm{dom}(\delta) \iff P \in I$

Let $\varphi_{p'}(m) = \begin{cases} \ulcorner \underline{n} \urcorner & \text{if } \varphi_p(m) = n \\ \uparrow & \text{otherwise} \end{cases}$, apply ADNT to get $\varphi_{p''} \in \mathcal{R}$.

'undefined' à la Visser

Proof.

$$\delta(\ulcorner P \urcorner) = \varepsilon \ulcorner Q \urcorner. \ \ (\exists \ulcorner M = N \urcorner \in U \ \ \lambda\beta + (P = Q) \vdash M = N)$$

δ is an anti diagonal function for $\mathbf{\Lambda}_\beta$ & $\ulcorner P \urcorner \notin \text{dom}(\delta) \iff P \in I$

Let $\varphi_{p'}(m) = \begin{cases} \ulcorner \underline{n} \urcorner & \text{if } \varphi_p(m) = n \\ \uparrow & \text{otherwise} \end{cases}$, apply ADNT to get $\varphi_{p''} \in \mathcal{R}$.

Choose λ-term Q representing $\psi(p, m) = \varphi_{p''}(m)$ in $\lambda\beta$.

'undefined' à la Visser

Proof.

$$\delta(\ulcorner P \urcorner) = \varepsilon \ulcorner Q \urcorner. \ \ (\exists \ulcorner M = N \urcorner \in U \ \ \lambda\beta + (P = Q) \vdash M = N)$$

δ is an anti diagonal function for $\mathbf{\Lambda}_\beta$ & $\ulcorner P \urcorner \notin \text{dom}(\delta) \iff P \in I$

Let $\varphi_{p'}(m) = \begin{cases} \ulcorner \underline{n} \urcorner & \text{if } \varphi_p(m) = n \\ \uparrow & \text{otherwise} \end{cases}$, apply ADNT to get $\varphi_{p''} \in \mathcal{R}$.

Choose λ-term Q representing $\psi(p, m) = \varphi_{p''}(m)$ in $\lambda\beta$.

Define $F = \lambda xy.\boldsymbol{E}(Qxy)$.

‘undefined’ à la Visser

Proof.

□

'undefined' à la Visser

Proof.

$$\varphi_p(m) = n$$

□

'undefined' à la Visser

Proof.

$$\varphi_p(m) = n \implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner$$

□

'undefined' à la Visser

Proof.

$$\varphi_p(m) = n \implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner$$
$$\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner$$

□

'undefined' à la Visser

Proof.

$$
\begin{aligned}
\varphi_p(m) = n &\implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner \\
&\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner \\
&\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner M \urcorner} = M = \underline{n}
\end{aligned}
$$

□

‘undefined’ à la Visser

Proof.

$$\begin{aligned}\varphi_p(m) = n &\implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner \\ &\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner \\ &\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner M \urcorner} = M = \underline{n}\end{aligned}$$

$$\varphi_p(m)\uparrow$$

□

'undefined' à la Visser

Proof.

$$\varphi_p(m) = n \implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner$$
$$\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner$$
$$\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner M \urcorner} = M = \underline{n}$$

$$\varphi_p(m)\uparrow \implies \varphi_{p'}\uparrow$$

□

'undefined' à la Visser

Proof.

$$\begin{aligned} \varphi_p(m) = n &\implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner \\ &\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner \\ &\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner M \urcorner} = M = \underline{n} \end{aligned}$$

$$\begin{aligned} \varphi_p(m)\uparrow &\implies \varphi_{p'}\uparrow \\ &\implies \varphi_{p''} = \ulcorner \Omega_0 \urcorner \end{aligned}$$

□

'undefined' à la Visser

Proof.

$$\begin{aligned}\varphi_p(m) = n &\implies \varphi_{p'}(m) = \ulcorner \underline{n} \urcorner \\ &\implies \varphi_{p''}(m) = \ulcorner M \urcorner \text{ where } \ulcorner M \urcorner \sim_E \ulcorner \underline{n} \urcorner \\ &\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner M \urcorner} = M = \underline{n}\end{aligned}$$

$$\begin{aligned}\varphi_p(m)\uparrow &\implies \varphi_{p'}\uparrow \\ &\implies \varphi_{p''} = \ulcorner \Omega_0 \urcorner \\ &\implies \lambda\beta \vdash \boldsymbol{E}(Q\ \underline{p}\ \underline{m}) = \boldsymbol{E}\underline{\ulcorner \Omega_0 \urcorner} = \Omega_0 \in I\end{aligned}$$

□

'undefined' à la Barendregt

Corollary of ADNT

Let γ be precomplete and $B \subseteq \mathbb{N}$ be a non-trivial r.e. set closed under $\sim_\gamma$.
Then $\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; f$ totalises ψ modulo $\sim_\gamma$ avoiding B.

‘undefined’ à la Barendregt

Corollary of ADNT

Let γ be precomplete and $B \subseteq \mathbb{N}$ be a non-trivial r.e. set closed under $\sim_\gamma$.
Then $\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; f$ totalises ψ modulo $\sim_\gamma$ avoiding B.

Proof.

Let $n_0 \notin B$.

□

'undefined' à la Barendregt

Corollary of ADNT

Let γ be precomplete and $B \subseteq \mathbb{N}$ be a non-trivial r.e. set closed under $\sim_\gamma$.
Then $\forall \psi \in \mathcal{PR} \ \ \exists f \in \mathcal{R} \ \ f$ totalises ψ modulo $\sim_\gamma$ avoiding B.

Proof.

Let $n_0 \notin B$. Define $\delta(n) = \begin{cases} n_0 & \text{if } n \in B \\ \uparrow & \text{otherwise} \end{cases}$.

□

'undefined' à la Barendregt

Corollary of ADNT

Let γ be precomplete and $B \subseteq \mathbb{N}$ be a non-trivial r.e. set closed under $\sim_\gamma$.
Then $\forall \psi \in \mathcal{PR} \;\; \exists f \in \mathcal{R} \;\; f$ totalises ψ modulo $\sim_\gamma$ avoiding B.

Proof.

Let $n_0 \notin B$. Define $\delta(n) = \begin{cases} n_0 & \text{if } n \in B \\ \uparrow & \text{otherwise} \end{cases}$.
Then $\delta(n)\downarrow \implies \delta(n) = n_0 \nsim_\gamma n$, i.e. δ is an anti diagonal function and the ADNT applies. □

'undefined' à la Barendregt

Definition of a Visser set

A set $\mathcal{B} \subseteq \Lambda^\circ$ is called a *Visser* set if

- $\mathcal{B}$ is r.e. (i.e. $\{\ulcorner M \urcorner \mid M \in \mathcal{B}\}$ is r.e.)
- $\mathcal{B}$ is closed under $=_\beta$

‘undefined’ à la Barendregt

Barendregt’s Theorem

Let $\mathcal{B} \subseteq \Lambda^\circ$ be a non-trivial Visser set. Then

$$\forall \psi \in \mathcal{PR} \quad \exists F \in \Lambda^\circ \quad \forall n \in \mathbb{N} \begin{cases} \psi(n)\downarrow \implies F\ \underline{n} =_\beta \underline{\psi(n)} \\ \psi(n)\uparrow \implies F\ \underline{n} \notin \mathcal{B} \end{cases}$$

'undefined' à la Barendregt

Proof.

'undefined' à la Barendregt

Proof.

Define $B = \{n \mid \boldsymbol{E}\ \underline{n} \in \mathcal{B}\}$.

'undefined' à la Barendregt

Proof.

Define $B = \{n \mid \boldsymbol{E}\ \underline{n} \in \mathcal{B}\}$.

Given $\psi \in \mathcal{PR}$ define $\psi_1(n) = \begin{cases} \ulcorner \underline{\psi(n)} \urcorner & \text{if } \psi(n)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$.

'undefined' à la Barendregt

Proof.

Define $B = \{n \mid \boldsymbol{E}\ \underline{n} \in \mathcal{B}\}$.

Given $\psi \in \mathcal{PR}$ define $\psi_1(n) = \begin{cases} \ulcorner \underline{\psi(n)} \urcorner & \text{if } \psi(n)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$.

Then $\boldsymbol{E}\ \underline{\psi_1(n)} =_\beta \underline{\psi(n)}$.

'undefined' à la Barendregt

Proof.

Define $B = \{n \mid \boldsymbol{E}\ \underline{n} \in \mathcal{B}\}$.

Given $\psi \in \mathcal{PR}$ define $\psi_1(n) = \begin{cases} \underline{\ulcorner \psi(n) \urcorner} & \text{if } \psi(n)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$.

Then $\boldsymbol{E}\ \underline{\psi_1(n)} =_\beta \underline{\psi(n)}$.
By the ADNT there exists an $f_1 \in \mathcal{R}$ that totalises ψ_1 modulo $\sim_E$ avoiding B.

'undefined' à la Barendregt

Proof.

Define $B = \{n \mid \boldsymbol{E}\ \underline{n} \in \mathcal{B}\}$.

Given $\psi \in \mathcal{PR}$ define $\psi_1(n) = \begin{cases} \ulcorner \underline{\psi(n)} \urcorner & \text{if } \psi(n)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$.

Then $\boldsymbol{E}\ \underline{\psi_1(n)} =_\beta \underline{\psi(n)}$.
By the $\overline{\text{ADNT}}$ there exists an $f_1 \in \mathcal{R}$ that totalises ψ_1 modulo $\sim_E$ avoiding B.
Let F_1 λ-define f_1.

'undefined' à la Barendregt

Proof.

□

'undefined' à la Barendregt

Proof.

$\psi(n)\downarrow$

□

‘undefined’ à la Barendregt

Proof.

$$\psi(n)\downarrow \implies \psi_1(n)\downarrow$$

□

'undefined' à la Barendregt

Proof.

$$\psi(n)\downarrow \implies \psi_1(n)\downarrow$$
$$\implies \boldsymbol{E} \circ F_1\ \underline{n} =_\beta \boldsymbol{E}\ \underline{f_1(n)} =_\beta \boldsymbol{E}\ \underline{\psi_1(n)} = \underline{\psi(n)}$$

□

'undefined' à la Barendregt

Proof.

$$\psi(n)\downarrow \implies \psi_1(n)\downarrow$$
$$\implies \boldsymbol{E} \circ F_1 \ \underline{n} =_\beta \boldsymbol{E} \ \underline{f_1(n)} =_\beta \boldsymbol{E} \ \underline{\psi_1(n)} = \underline{\psi(n)}$$

$$\psi(n)\uparrow \implies \psi_1(n)\uparrow$$

□

‘undefined’ à la Barendregt

Proof.

$$\begin{aligned}\psi(n)\downarrow &\implies \psi_1(n)\downarrow \\ &\implies \mathbf{E}\circ F_1\ \underline{n} =_\beta \mathbf{E}\ \underline{f_1(n)} =_\beta \mathbf{E}\ \underline{\psi_1(n)} = \underline{\psi(n)}\end{aligned}$$

$$\begin{aligned}\psi(n)\uparrow &\implies \psi_1(n)\uparrow \\ &\implies f_1(n) \notin B\end{aligned}$$

□

'undefined' à la Barendregt

Proof.

$$\begin{aligned} \psi(n)\downarrow &\implies \psi_1(n)\downarrow \\ &\implies \mathbf{E}\circ F_1\ \underline{n} =_\beta \mathbf{E}\ \underline{f_1(n)} =_\beta \mathbf{E}\ \underline{\psi_1(n)} = \underline{\psi(n)} \end{aligned}$$

$$\begin{aligned} \psi(n)\uparrow &\implies \psi_1(n)\uparrow \\ &\implies f_1(n)\notin B \\ &\implies \mathbf{E}\ \underline{f_1(n)} \notin \mathcal{B} \end{aligned}$$

□

'undefined' à la Barendregt

Proof.

$$
\begin{aligned}
\psi(n)\downarrow &\implies \psi_1(n)\downarrow \\
&\implies \boldsymbol{E} \circ F_1\ \underline{n} =_\beta \boldsymbol{E}\ \underline{f_1(n)} =_\beta \boldsymbol{E}\ \underline{\psi_1(n)} = \underline{\psi(n)}
\end{aligned}
$$

$$
\begin{aligned}
\psi(n)\uparrow &\implies \psi_1(n)\uparrow \\
&\implies f_1(n) \notin B \\
&\implies \boldsymbol{E}\ \underline{f_1(n)} \notin \mathcal{B} \\
&\implies \boldsymbol{E} \circ F_1\ \underline{n} \notin \mathcal{B}
\end{aligned}
$$

□

'undefined' à la Barendregt

Proof.

$$\begin{aligned} \psi(n)\downarrow &\implies \psi_1(n)\downarrow \\ &\implies \boldsymbol{E}\circ F_1\ \underline{n} =_\beta \boldsymbol{E}\ \underline{f_1(n)} =_\beta \boldsymbol{E}\ \underline{\psi_1(n)} = \underline{\psi(n)} \end{aligned}$$

$$\begin{aligned} \psi(n)\uparrow &\implies \psi_1(n)\uparrow \\ &\implies f_1(n) \notin B \\ &\implies \boldsymbol{E}\ \underline{f_1(n)} \notin \mathcal{B} \\ &\implies \boldsymbol{E}\circ F_1\ \underline{n} \notin \mathcal{B} \end{aligned}$$

Take $F = \boldsymbol{E}\circ F_1$. □

'undefined' à la Barendregt

Corollary

Let $\mathcal{A} \subseteq \Lambda^\circ$ be one of the following sets:

1. $\mathcal{A} = \{M \in \Lambda^\circ \mid M \text{ has no normal form}\}$
2. $\mathcal{A} = \{M \in \Lambda^\circ \mid M \text{ is unsolvable}\}$
3. $\mathcal{A} = \{M \in \Lambda^\circ \mid M \text{ is easy}\}$
4. $\mathcal{A} = \{M \in \Lambda^\circ \mid M \text{ is of order } 0\}$

Then every $\psi \in \mathcal{PR}$ can be λ-defined by an $F \in \Lambda^\circ$ such that

$$\psi(n)\downarrow \implies F\ \underline{n} =_\beta \underline{\psi(n)}$$

$$\psi(n)\uparrow \implies F\ \underline{n} \in \mathcal{A}$$

Say all $\psi \in \mathcal{PR}$ can be λ-defined w.r.t. $\mathcal{A}$ as undefined elements.

Fin