Mathematical Foundations of Computer Science

Decidability of String Graphs

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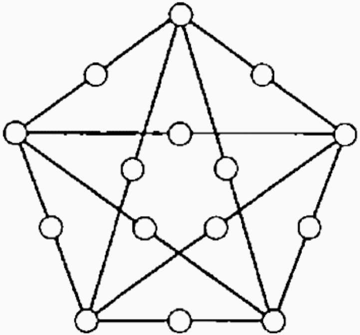
Introduction

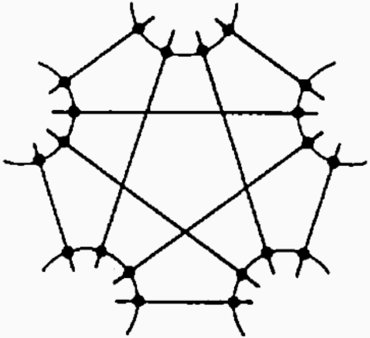
What is a string graph? [SŠ01]

- A *curve* (or *string*) is a set homeomorphic to $[0, 1]$
- Given a collection of curves $(C_i)_{i \in I}$ in the plane, the corresponding intersection graph is $(I, \{\{i, j\} : C_i \text{ and } C_j \text{ intersect}\})$
- *string graph*: a graph isomorphic to the intersection graph of a collection of curves

Non-string graph example [Sin66]

There are graph instances that cannot be string graphs.





Definitions

- *size* of a collection of curves: the number of intersection points
- $c_s(G)$: the size of the smallest (smallest number of intersections) set of curves whose intersection graph is isomorphic to G
- Define $c_s(m) = \max\{c_s(G) : G \text{ has } m \text{ edges}\}$

$c_s(G)$ is finite [SŠ01]

- It is not obvious that $c_s(G)$ is finite for every string graph G
- Kratochvíl et al. showed that $c_s(G)$ is finite for every string graph G

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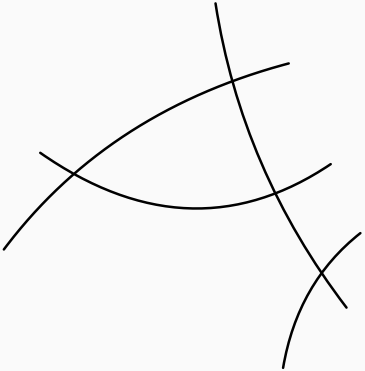

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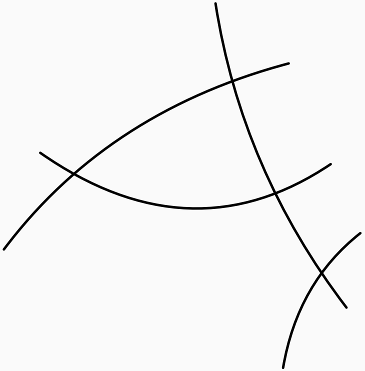

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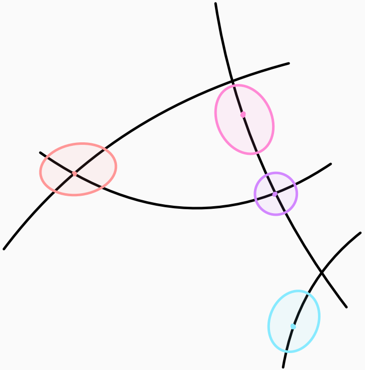

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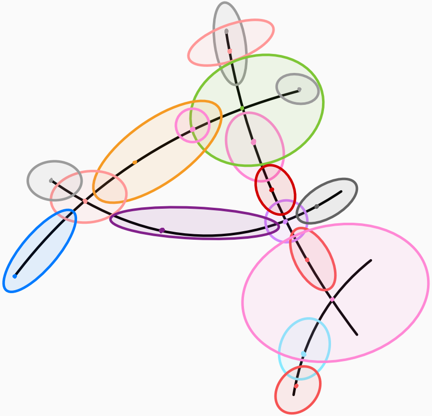
- $\mathcal{O} := \{O_p : p \in C\}$ is an open cover of C.



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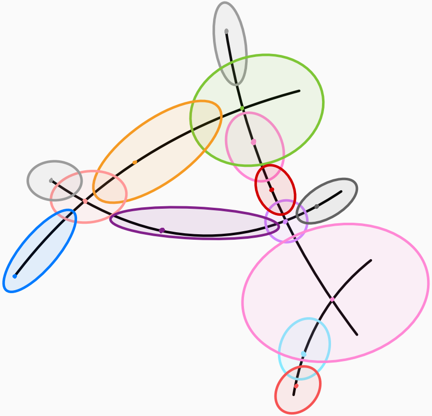

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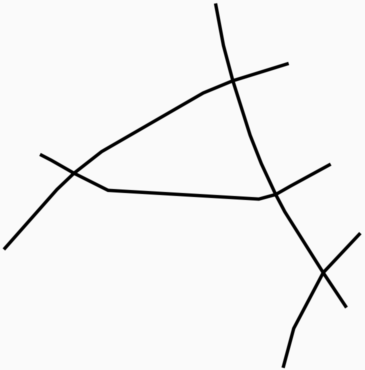
- In each $O \in \mathcal{O}'$, replace the curves by polygonal arcs. $\square$



Realizability

- Graph $G = (V, E)$, $R \subseteq \binom{E}{2} = \{\{e, f\} : e, f \in E\}$ on E
- call a drawing D of G in the plane a *weak realization* of (G, R) if only pairs of edges $\in R$ are allowed to intersect in D
- call a drawing D of G in the plane a *realization* of (G, R) if exactly pairs of edges $\in R$ are intersect in D

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- $c_w(G, R)$ is the smallest number of intersections in a weak realization of (G, R)
- $c_w(G) = \max\{c_w(G, R) : (G, R)$ has a weak realization$\}$
- $c_w(m) = \max\{c_w(G) : G$ has m edges$\}$
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- Define $c_r(G, R)$, $c_r(G)$ and $c_r(m)$ similarly for realizations
- $\mathrm{cr}(G) = c_w(G, \binom{E}{2})$ is known as the crossing number (**NP**-complete)
- $c_w(G, \emptyset)$ is equivalent to planarity testing (in **P**)

Relationships between realizability functions [SŠ01] [Kra91]

Lemma 2
c_r, c_w and c_s are equivalent in the following sense:

- $c_w(m) \leq c_r(m)$
- $c_r(m) \leq 4c_s(m^2 + 4m)$
- $c_s(m) \leq 4c_w(2m) + 2m$

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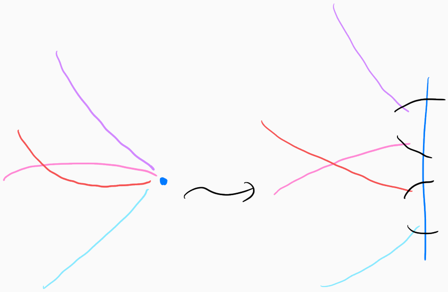
$V' = V \cup E \cup \{(u, e) : u \in e \in E\}$ and
$E' = R \cup \{\{u, (u, e)\}, u \in e \in E\} \cup \{\{e, (u, e)\}, u \in e \in E\}$.

Let $G = (V, E)$, $R \subseteq \binom{E}{2}$ be a realizability problem. Construct $V' = V \cup E \cup \{(u, e) : u \in e \in E\}$ and $E' = R \cup \{\{u, (u, e)\}, u \in e \in E\} \cup \{\{e, (u, e)\}, u \in e \in E\}$. Then (G, R) is realizable iff $G' = (V', E')$ is a string graph.



String $\propto$ Weak Realizability

Let $G = (V, E)$ be a string graph instance.

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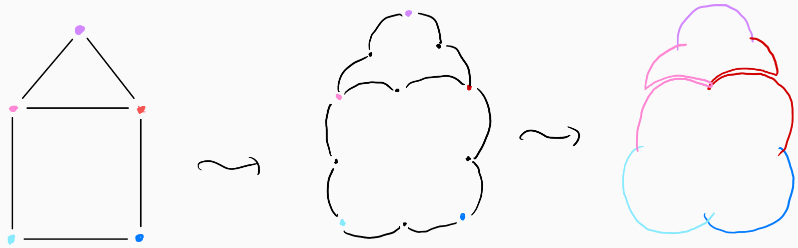
$G' = (V \cup E, \{\{u, e\} : u \in e \in E\})$ and
$R = \{\{\{u, e\}, \{v, f\}\} : \{u, v\} \in E\}$.



Bounding the Number of Intersections [SŠ01]

Bounding the Number of Intersections – Strategy

Show by contradiction: There exists a drawing for a realizability problem s.t. there are $< 2^m$ intersections for each edge e in D.

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Every string graph must have a representation with maximally exponential size.

Construct an algorithm to check if something is a string graph in **NEXP**.

Bounding the Number of Intersections 1

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Every word of length $\geq 2^n$ over an alphabet of size n contains a non-trivial (contiguous) subword in which every character occurs an even number of times.

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- Let $\Sigma = \{1, \ldots, n\}$, $w \in \Sigma^*, |w| \geq 2^n$.
- $\forall i \in \{0, \ldots, 2^n\}$, assign a vector $v_i \in \mathbb{Z}_2^n$ whose j-th coordinate $=$ parity of #occurences of symbol j in the $w[: i]$.

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$$v_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_6 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_8 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

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- $\exists 2^n + 1$ indices $\Rightarrow$ Pigeonhole Principle $\Rightarrow \exists i < j$ with $v_i = v_j$.
- For $w' := w[(i+1) : j]$, every symbol occurs an even number of times. □

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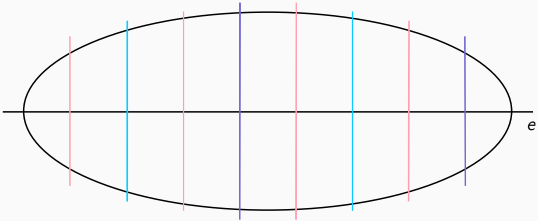
$$w' = w[3:8] = 312231$$

Bounding the Number of Intersections 2

Theorem 4
Let G be a graph with m edges, $R \subseteq \binom{E}{2}$ such that (G, R) is weakly realizable, and let D be a weak realization of (G, R) with the minimal number of intersections. Then for any edge $e \in G$ there are less than 2^m intersections on the curve realizing e in D.

Theorem 4 Proof

- Suppose not. Let D be a weak realization of (G, R) with minimal number of intersections.



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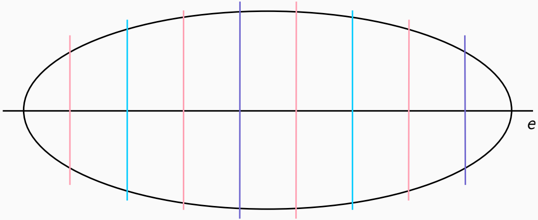

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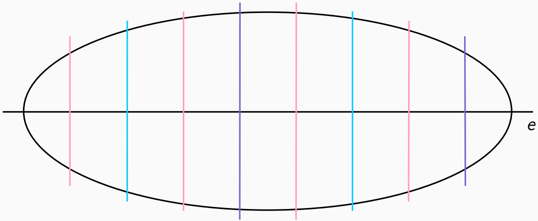
- Suppose not. Let D be a weak realization of (G, R) with minimal number of intersections.
- Let e be an edge which has $\geq 2^m$ intersections.
- Choose a segment of e which is intersected an even number of times by any other edge. Draw a window around this segment with no other intersections.



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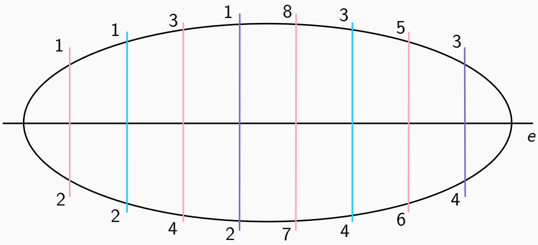
- Let $2n_f$ be the number of intersections of edge f with e in this window.



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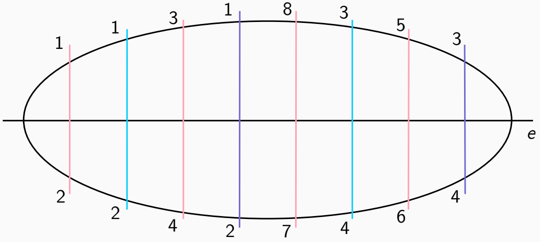
- Let $2n_f$ be the number of intersections of edge f with e in this window.
- For each f, assign numbers $1, \ldots, 4n_f$ to the intersections with the window in the order they appear on f.



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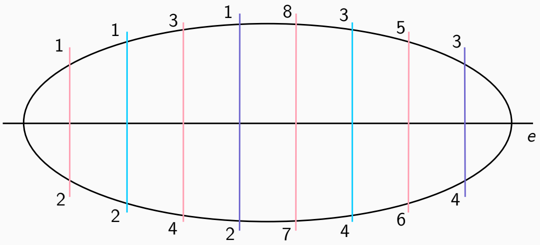
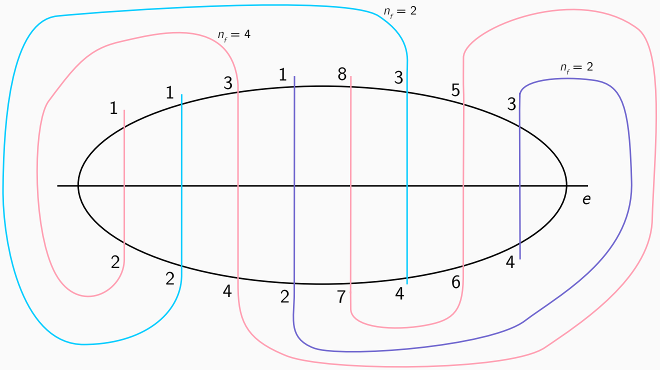
- Let $2n_f$ be the number of intersections of edge f with e in this window.
- For each f, assign numbers $1, \ldots, 4n_f$ to the intersections with the window in the order they appear on f.
- We can assume (by Jordan-Schoenflies theorem) that the window is a circle and e is a straight line and $\forall f$, the intersections $2i - 1$ and $2i$ are mirror images.



Theorem 4 Proof

- For each f, there is a connection between $4i - 2$ and $4i - 1$ ($i \in \{1, \ldots, n_f\}$) lying completely outside the window.



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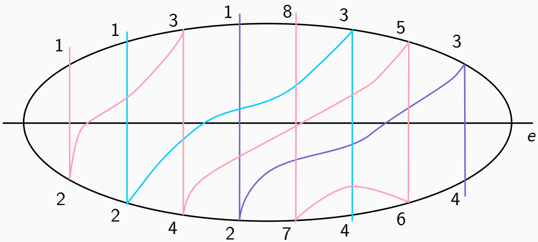
- Use circular inversion along the circle to bring all of these conections inside.



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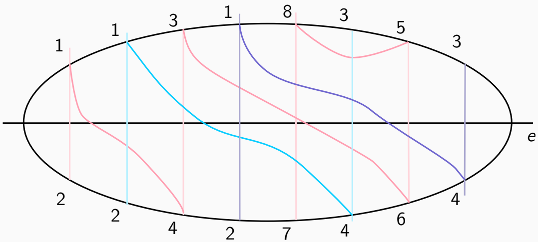
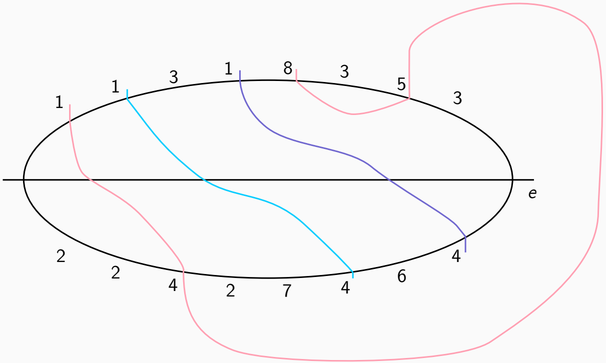
- Mirror everything inside of the window along e.



- Now, we have for every edge a connection between $4i-3$ and $4i$ which is inside the window.



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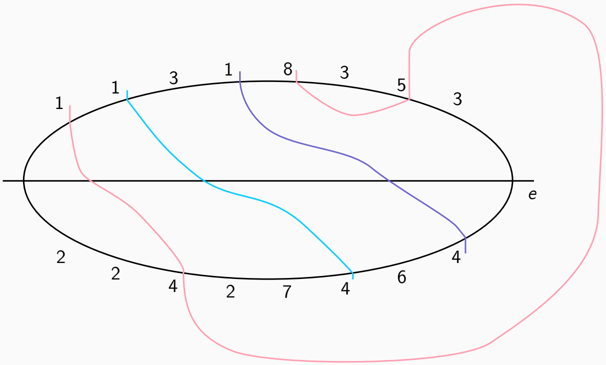
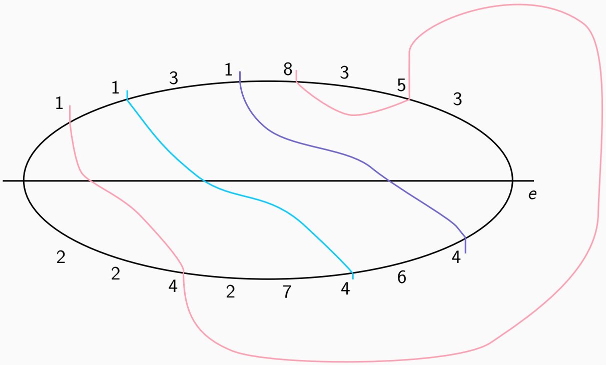
- Now, we have for every edge a connection between $4i-3$ and $4i$ which is inside the window.
- Build a new version f': start at intersection 1 (connected to one of the endpoints of f), continue to 4 (inside), 5 (outside), etc. until $4n_f$.



Theorem 4 Proof

- f intersects e an even number of times $\Rightarrow$ f' still connects the two original endpoints.



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- f intersects e an even number of times $\Rightarrow f'$ still connects the two original endpoints.
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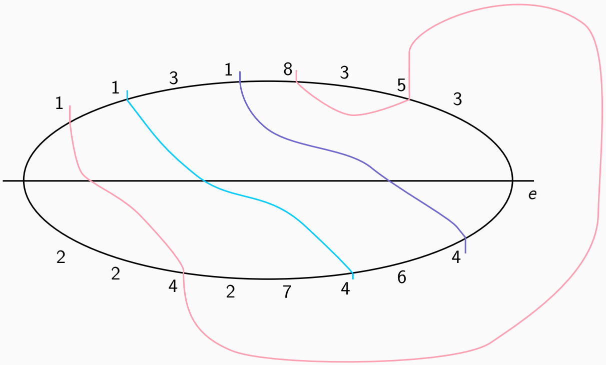

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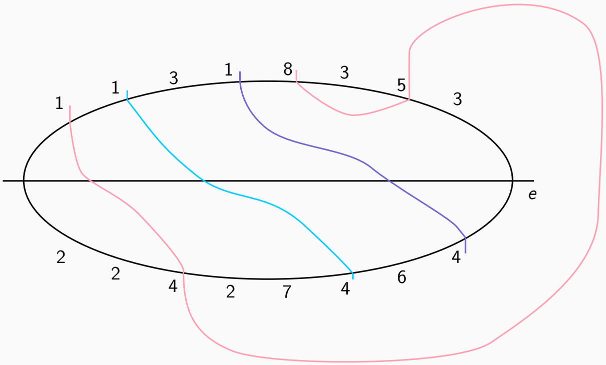
- $\rightsquigarrow$ reduced the number of intersections with the window from $4n_f$ to $2n_f$.
- Every intersection between the curves inside the circle corresponds to an intersection outside $\Rightarrow$ the new realization respects R (in this example, there are no intersections).



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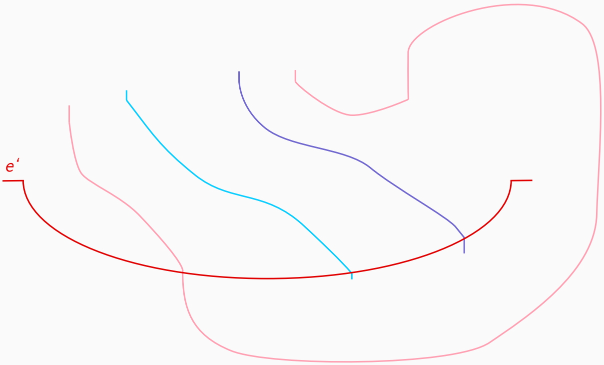

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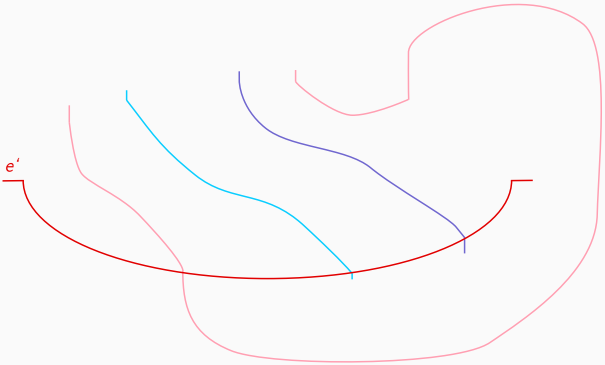
- f' halved the number of intersections between the f' and the window boundary.



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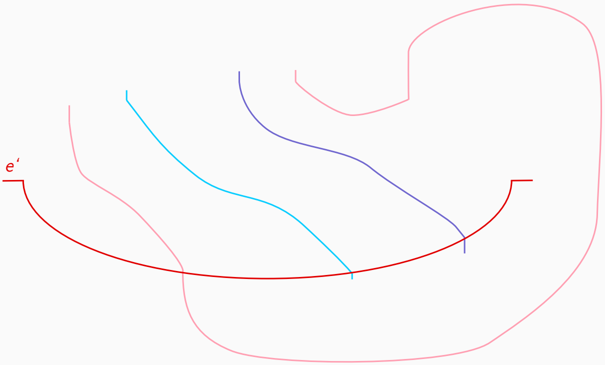
- The number of intersections along e might have increased.
- f' halved the number of intersections between the f' and the window boundary.
- $\rightsquigarrow$ $e' :=$ one of the two sides of the boundary of the window (at least one side has less intersections than before). □



String graph decidability

Corollary 5
String graph recognition is in **NEXP**.

Theorem 6 (Schnyder)
Each plane graph with $n \geq 3$ *vertices has a straight line embedding on the* $n-2$ *by* $n-2$ *grid.*

Corollary 5 Proof

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- Let G be a string graph with m vertices.

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- Using Theorem 4, there must be a drawing with $\leq 2^{cm}$ intersections per edge.
  $\rightsquigarrow$ The collection of curves also has $M \leq 2^{\tilde{c}m}$ intersections.

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- Consider this collection of curves of size M and draw them as a plane graph (intersection $\mapsto$ vertex), at most M vertices.
- By Theorem 6, there is a drawing of this graph on an $(M-2) \times (M-2)$ grid.
- $\Rightarrow$ in **NEXP**, guess a graph ($\sim$ collection of curves) on such a grid and verify whether its intersection graph is isomorphic to G. $\square$

String Graphs Requiring Exponential Representations [KM91]

String Graphs Requiring Exponential Representations

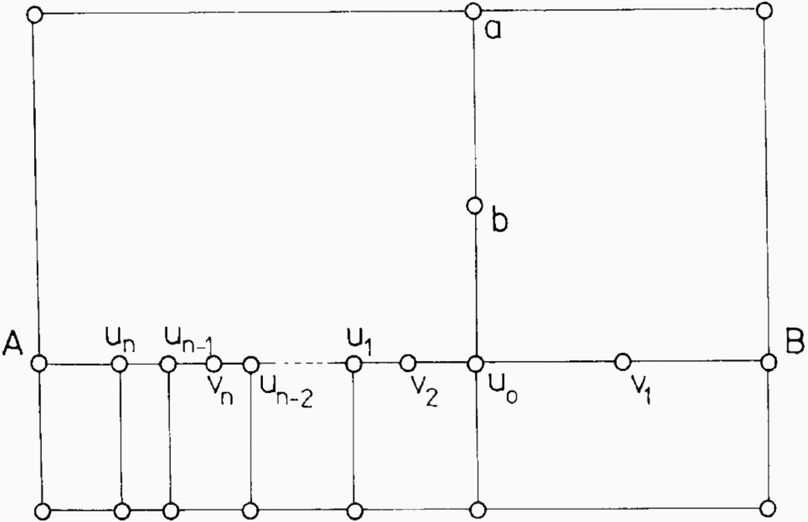
Goal: provide a construction of a graph on $O(n)$ vertices which can be represented as a string graph but every realization of the graph requires an exponential number of intersections.

$\rightsquigarrow$ for string graph testing, we need to check at least exponentially many realizations

Exponential Representations

Theorem 7
$c_w(m) \geq 2^{cm}$ *for some constant* $c > 0$.

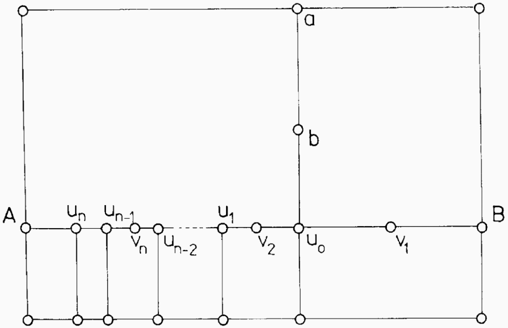
Theorem 7 Proof



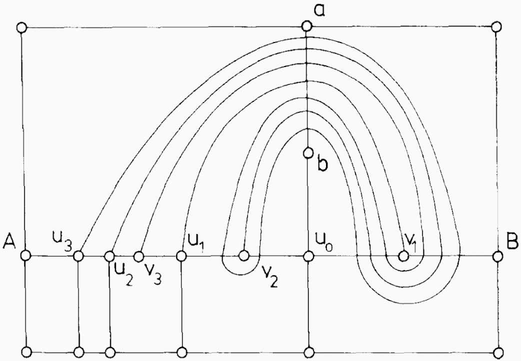
Theorem 7 Proof

Theorem 7
$c_w(m) \geq 2^{cm}$ for some constant $c > 0$.

- This graph has a topologically unique drawing.
- Add edges $\{u_i, v_i\}, i = 1, 2, \ldots, n$, allow them to cross only the edge $\{a, b\}$ and the horizontal path AB, not each other.



Lemma
In every weak realization, the edge $\{u_i, v_i\}$ *intersects the edge* $\{a, b\}$ *at least* 2^{i-1} *times.*



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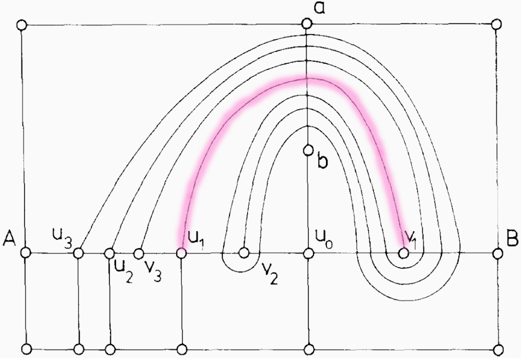
- Induction start, $i = 1$: $2^{1-0} = 1$ exactly one crossing of $\{a, b\}$ ✓



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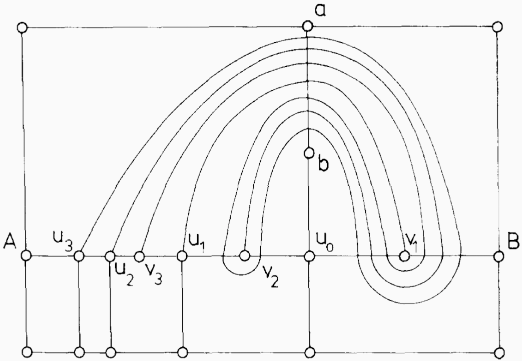

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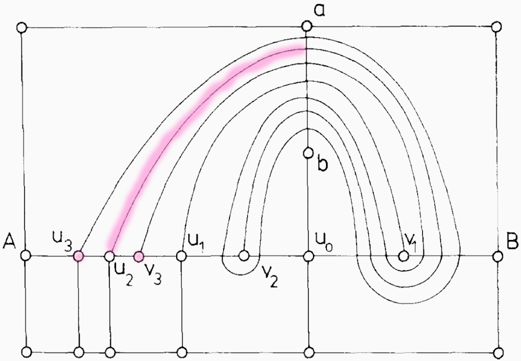
- $\{u_{i+1}, v_{i+1}\}$ is not allowed to cross $\{u_i, v_i\}$.



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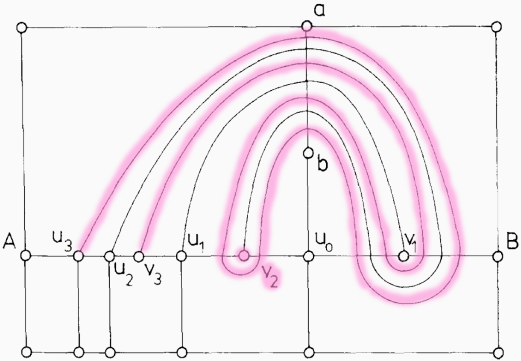
- Induction step: Let it hold for some i.
- $\{u_{i+1}, v_{i+1}\}$ is not allowed to cross $\{u_i, v_i\}$.
- $\Rightarrow$ it must make a turn around v_i.



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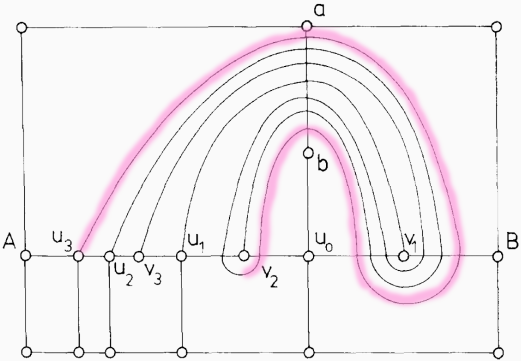
- Thus, it consists of two parts:
    1. from u_{i+1} to the turnpoint



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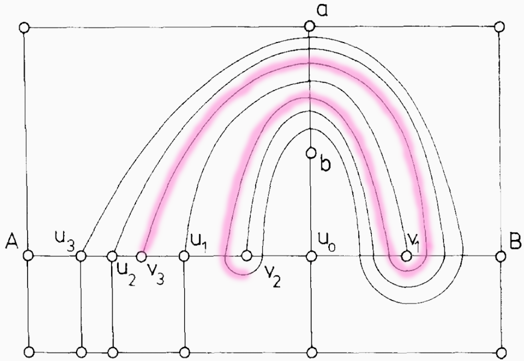
  2. from the turnpoint to v_{i+1}



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In every weak realization, the edge $\{u_i, v_i\}$ intersects the edge $\{a, b\}$ at least 2^{i-1} times.

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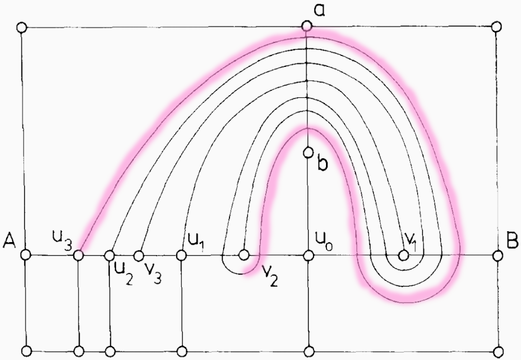

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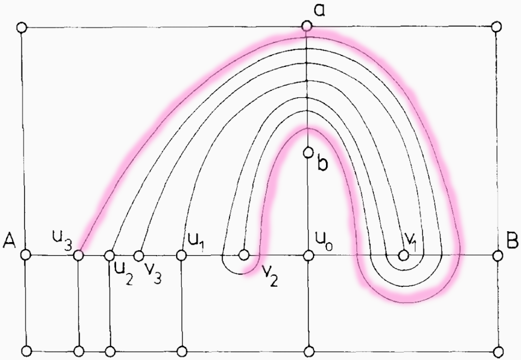

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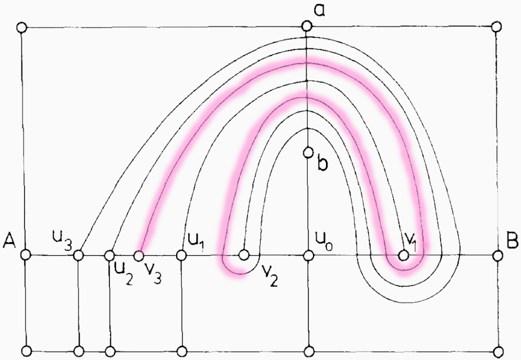

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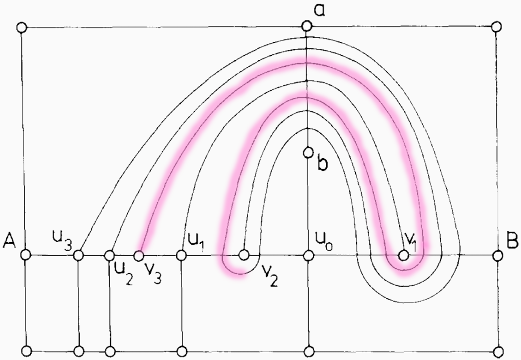

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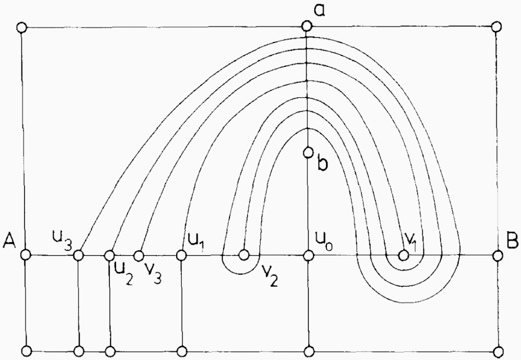
- $\Rightarrow 2^{i-1} + 2^{i-1} = 2^i = 2^{(i+1)-1}$ intersections ✓



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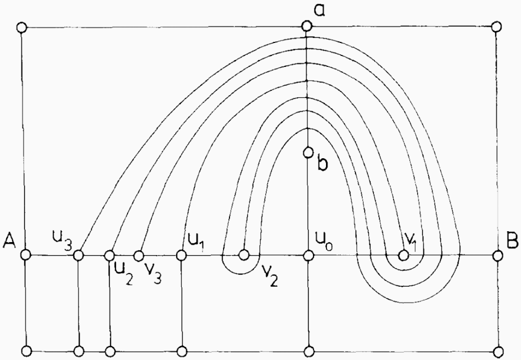
- $\Rightarrow 2^{i-1} + 2^{i-1} = 2^i = 2^{(i+1)-1}$ intersections ✓
- $\Rightarrow c_w(5n + 13) \geq 2^{n-1}$. □



Exponential Representations

Corollary 8
$c_s(m) \geq 2^{\hat{c}m}$ *for some constant* $\hat{c} > 0$.

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