Linear Lambda Calculus is Linear

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Papers

Díaz-Caro A. and Dowek G.
"Linear lambda-calculus is linear"

Vaux, L.
"The algebraic lambda calculus"

Outline

Linear Logic

A logic is called linear if each hypothesis is used exactly once. Similarly, a lambda calculus is called linear if all variables are used exactly once.

Linear Logic

A logic is called linear if each hypothesis is used exactly once. Similarly, a lambda calculus is called linear if all variables are used exactly once.

Definition (A Simple Linear Type Theory)

Variable Rule: $$\frac{}{x : A \vdash x : A}\ \text{ax}$$

Abstraction: $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : A \Rightarrow B} \Rightarrow\text{-i}$$

Application: $$\frac{\Gamma \vdash t : A \Rightarrow B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\, u : B} \Rightarrow\text{-e}$$

This differs from Simple Type Theory in 2 places:

- In the variable rule, $x : A$ must be the only item in the context.
- In the application rule, each variable has to be used in either t or u, but not both.

Linear Logic

Definition (Linear Conjunction Attempt 1)

Derive the two elements in the conjunction from different contexts.

$$\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma, \Delta \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \delta^1_\wedge(t) : A} \wedge\text{-e1} \quad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \delta^2_\wedge(t) : B} \wedge\text{-e2}.$$

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We want to combine this with the following reductions:

$$\delta^1_\wedge(\langle t, u \rangle) \to t$$

$$\delta^2_\wedge(\langle t, u \rangle) \to u.$$

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We want to combine this with the following reductions:

$$\delta^1_\wedge(\langle t, u \rangle) \to t$$
$$\delta^2_\wedge(\langle t, u \rangle) \to u.$$

Problem with Subject Reduction

If $t : A \vdash t : A$ and $u : B \vdash u : B$. Then:
$t : A, u : B \vdash \delta^1_\wedge(\langle t, u \rangle) : A$, and $\delta^1_\wedge(\langle t, u \rangle) \to t$, but $t : A, u : B \nvdash t : A$.

Linear Logic

Definition (Additive Conjunction[1])

Derive the two elements in the conjunction from the same context.

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \delta^1_\wedge(t) : A} \wedge\text{-e1} \quad \frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash \delta^2_\wedge(t) : B} \wedge\text{-e2}.$$

Now the subject reduction holds for the following reductions.

$$\delta^1_\wedge(\langle t, u \rangle) \to t$$
$$\delta^2_\wedge(\langle t, u \rangle) \to u.$$

[1] A linear multiplicative conjunction can be defined using an elimination that takes a function of two arguments.

Example not derivable in Linear Logic

Example

The following proposition is derivable in propositional logic, but not the linear logic that we defined:

$$A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C$$

Proof.

$$\vdash A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C$$

□

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Proof.

$$\cfrac{\cfrac{x : A \wedge B, y : A \Rightarrow B \Rightarrow C \vdash C}{x : A \wedge B \vdash (A \Rightarrow B \Rightarrow C) \Rightarrow C} \Rightarrow\text{-i}}{\vdash A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C} \Rightarrow\text{-i}$$

□

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Proof.

$$\dfrac{\dfrac{\dfrac{x : A \wedge B, y : A \Rightarrow B \Rightarrow C \vdash B \Rightarrow C \quad \vdash A}{x : A \wedge B, y : A \Rightarrow B \Rightarrow C \vdash C} \Rightarrow\text{-e}}{x : A \wedge B \vdash (A \Rightarrow B \Rightarrow C) \Rightarrow C} \Rightarrow\text{-i}}{\vdash A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C} \Rightarrow\text{-i}$$

□

Example not derivable in Linear Logic

Example

The following proposition is derivable in propositional logic, but not the linear logic that we defined:

$$A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : A \Rightarrow B \Rightarrow C \vdash A \Rightarrow B \Rightarrow C}\ \mathit{ax} \quad \dfrac{\dfrac{}{x : A \wedge B \vdash A \wedge B}\ \mathit{ax}}{x : A \wedge B \vdash A}\ \wedge\text{-e1}}{x : A \wedge B, y : A \Rightarrow B \Rightarrow C \vdash B \Rightarrow C}\ \Rightarrow\text{-e} \quad \color{red}{\vdash A}}{\dfrac{x : A \wedge B, y : A \Rightarrow B \Rightarrow C \vdash C}{x : A \wedge B \vdash (A \Rightarrow B \Rightarrow C) \Rightarrow C}\ \Rightarrow\text{-i}}\ \Rightarrow\text{-e}}{\vdash A \wedge B \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C}\ \Rightarrow\text{-i}$$

□

Linear Logic

The name linear logic suggests that linear proofs from A to B should represent linear functions.
However, the simple linear type theory does define addition and scalar multiplication.

Definition (Algebraic Linearity)

A function $f : X \to Y$ between vector spaces X and Y is called linear if for all $x, y \in X$ and scalars $a \in \mathcal{S}$ in field $\mathcal{S}$:

$$f(x + y) = f(x) + f(y)$$
$$f(a \cdot x) = a \cdot f(x).$$

By adding rules for addition and multiplication, linearity in this linear logic can be proven between certain types of propositions (corresponding to vectors).

Conjunction in Linear Lambda Calculus

Definition (Additive Conjunction in Linear Lambda Calculus)

The conjunction in the Linear Lambda Calculus adds a λ-term as a second argument for the conjunction elimination:

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$
$$\frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash \delta^1_\wedge(t, x.u) : C} \wedge\text{-e1}$$
$$\frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : B \vdash u : C}{\Gamma \vdash \delta^2_\wedge(t, x.u) : C} \wedge\text{-e2} .$$

With the following reductions:

$$\delta^1_\wedge(\langle t, u \rangle, x.v) \to v[x := t]$$
$$\delta^2_\wedge(\langle t, u \rangle, x.v) \to v[x := u].$$

Outline

Interstitial Rules

The following rules are introduced for the sum and scalar multiplication:

Definition (Interstitial Rules)

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t + u : A}\ sum \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash a \bullet t : A}\ prod(a)$$

For scalars $a \in \mathcal{S}$ in a field $\mathcal{S}$.

These can be used to build proofs that cannot be reduced, as the introduction is not immediately followed by the elimination:

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\pi_1}{\Gamma \vdash A} \quad \cfrac{\pi_2}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-i} \quad \cfrac{\cfrac{\pi_3}{\Gamma \vdash A} \quad \cfrac{\pi_4}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\ \wedge\text{-i}}{\Gamma \vdash A \wedge B}\ \text{sum}}{\Gamma \vdash A}\ \wedge\text{-e1}$$

Scalars

In 1st-order propositional logic, we can create a term of type $\top$ in any context using $\top$-introduction.
In the Linear Lambda Calculus, we create a $\top$ introduction for each scalar $a \in \mathcal{S}$.

Definition ($\top$-introduction)

$$\frac{}{\vdash a.\star : \top}\ \top\text{-i}(a)$$

For scalars $a \in \mathcal{S}$ in a field $\mathcal{S}$.

Deduction Rules for Linear Lambda Calculus 1

$$\frac{}{x : A \vdash x : A}\ \text{ax} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t \boldsymbol{+} u : A}\ \text{sum} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash a \bullet t : A}\ \text{prod}(a)$$

$$\frac{}{\vdash a.\star : \top}\ \top\text{-i}(a) \qquad \frac{\Gamma \vdash t : \top \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash \delta_\top(t, u) : A}\ \top\text{-e} \qquad \frac{\Gamma \vdash t : \bot}{\Gamma, \Delta \vdash \delta_\bot(t) : C}\ \bot\text{-e}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ \Rightarrow\text{-i} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\, u : B}\ \Rightarrow\text{-e}$$

Deduction Rules for Linear Lambda Calculus 2

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \wedge B} \wedge\text{-i}$$

$$\frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : A \vdash u : C}{\Gamma, \Delta \vdash \delta^1_\wedge(t, x.u) : C} \wedge\text{-e1}$$

$$\frac{\Gamma \vdash t : A \wedge B \quad \Delta, x : B \vdash u : C}{\Gamma, \Delta \vdash \delta^2_\wedge(t, x.u) : C} \wedge\text{-e2}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash inl(t) : A \vee B} \vee\text{-i1} \quad \frac{\Gamma \vdash t : B}{\Gamma \vdash inr(t) : A \vee B} \vee\text{-i2}$$

$$\frac{\Gamma \vdash t : A \vee B \quad \Delta, x : A \vdash u : C \quad \Delta, y : B \vdash v : C}{\Gamma, \Delta \vdash \delta_\vee(t, x.u, y.v) : C} \vee\text{-e}$$

Reduction Rules for Linear Lambda Calculus 1

$$(\lambda x.t)u \to t[x := u]$$
$$\delta_\top(a.\star, t) \to a \bullet t$$

Reduction Rules for Linear Lambda Calculus 1

$$(\lambda x.t)u \to t[x := u]$$
$$\delta_\top(a.\star, t) \to a \bullet t$$

$$\delta^1_\wedge(\langle t, u\rangle, x.v) \to v[x := t]$$
$$\delta^2_\wedge(\langle t, u\rangle, x.v) \to v[x := u]$$

$$\delta_\vee(inl(t), x.u, y.v) \to u[x := t]$$
$$\delta_\vee(inr(t), x.u, y.v) \to v[y := t]$$

Reduction Rules for Linear Lambda Calculus 2

$$a. \star \mathbf{+} b.\star \to (a + b).\star$$
$$(\lambda x.t) \mathbf{+} (\lambda x.u) \to (\lambda x.t \mathbf{+} u)$$
$$\langle t, u\rangle \mathbf{+} \langle v, w\rangle \to \langle t \mathbf{+} v, u \mathbf{+} w\rangle$$
$$\delta_\vee(t \mathbf{+} u, x.v, y.w) \to \delta_\vee(t, x.v, y.w) \mathbf{+} \delta_\vee(u, x.v, y.w)$$

The third rule requires that conjunction $\wedge$ is additive, otherwise, t and v may not be derived from the same context.
The fourth rule is oriented the other way, this allows us to reduce $t = inl(t')$ and $u = inr(u')$.

Reduction Rules for Linear Lambda Calculus 2

$$
\begin{aligned}
a.\star \mathbin{\boldsymbol{+}} b.\star &\to (a+b).\star \\
(\lambda x.t) \mathbin{\boldsymbol{+}} (\lambda x.u) &\to (\lambda x.t \mathbin{\boldsymbol{+}} u) \\
\langle t, u\rangle \mathbin{\boldsymbol{+}} \langle v, w\rangle &\to \langle t \mathbin{\boldsymbol{+}} v, u \mathbin{\boldsymbol{+}} w\rangle \\
\delta_\vee(t \mathbin{\boldsymbol{+}} u, x.v, y.w) &\to \delta_\vee(t, x.v, y.w) \mathbin{\boldsymbol{+}} \delta_\vee(u, x.v, y.w)
\end{aligned}
$$

$$
\begin{aligned}
a \bullet b.\star &\to (a \times b).\star \\
a \bullet (\lambda x.t) &\to \lambda x.a \bullet t \\
a \bullet \langle t, u\rangle &\to \langle a \bullet t, a \bullet u\rangle \\
\delta_\vee(a \bullet t, x.v, y.w) &\to a \bullet \delta_\vee(t, x.v, y.w)
\end{aligned}
$$

Example Reduction

Example

$$\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (\delta_\vee(inl(3.\star), y.y, z.z)), \delta^2_\wedge(x, y.y))$$

Example Reduction

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$$\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (\delta_\vee(inl(3.\star), y.y, z.z)), \delta^2_\wedge(x, y.y))$$
$$\to \lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (3.\star), \delta^2_\wedge(x, y.y))$$

Example Reduction

Example

$$
\begin{aligned}
&\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (\delta_\vee(inl(3.\star), y.y, z.z)), \delta^2_\wedge(x, y.y)) \\
\to &\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (3.\star), \delta^2_\wedge(x, y.y)) \\
\to &\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(6.\star, \delta^2_\wedge(x, y.y))
\end{aligned}
$$

Example Reduction

Example

$$\lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (\delta_\vee(inl(3.\star), y.y, z.z)), \delta^2_\wedge(x, y.y))$$
$$\to \lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(2 \bullet (3.\star), \delta^2_\wedge(x, y.y))$$
$$\to \lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet \delta_\top(6.\star, \delta^2_\wedge(x, y.y))$$
$$\to \lambda x.\delta^1_\wedge(x, y.y) + 3 \bullet 6 \bullet \delta^2_\wedge(x, y.y)$$

Reduction Properties

Theorem (Subject Reduction)

If $\Gamma \vdash t : A$ and $t \to u$ then $\Gamma \vdash u : A$.

Theorem (Confluence)

For any term t, if $t \to^* u$ and $t \to^* v$, then there is a w such that $u \to^* w$ and $v \to^* w$.

Theorem (Strongly Terminating)

There is no infinite sequence of reductions $t \to t_1 \to t_2 \to \dots$ for any term t.

From confluence and strong termination, all terms have a unique normal form.

Reduction Properties

Theorem (Introduction)

Let t be a closed irreducible proof of type A. Then:

- *If $A = \top$, then t has the form $a.\star$*
- *The proposition A is not $\bot$*
- *If $A = B \Rightarrow C$ then t has the form $\lambda x.u$*
- *If $A = B \wedge C$ then t has the form $\langle u, v \rangle$*
- *If $A = B \vee C$ then t has the form $inl(u)$, $inr(u)$, $u + v$, or $a \bullet u$.*

Outline

Linearity of Linear Lambda Calculus

The aim is to show that:

1. All linear functions can be encoded as proofs of a specific form.
2. All proofs of a specific form are linear in the algebraic sense.

But for this it is necessary to define the specific form. We define this specific form to be closed proofs between two vector types $A, B \in \mathcal{V}$.

Vectors

There is a 1-to-1 correspondence between scalars $a \in \mathcal{S}$ and closed irreducible proofs $a.\star : \top$.

A similar 1-to-1 correspondence exists between $(a, b, c) \in \mathcal{S}^3$ and closed irreducible proofs $\langle\langle a.\star, b.\star\rangle, c.\star\rangle : (\top \wedge \top) \wedge \top$,

And also between $(a, b, c) \in \mathcal{S}^3$ and closed irreducible proofs $\langle a.\star, \langle b.\star, c.\star\rangle\rangle : \top \wedge (\top \wedge \top)$

Vectors

Definition (The set $\mathcal{V}$)

The set $\mathcal{V}$ is inductively defined as:

- $\top \in \mathcal{V}$
- If $A, B \in \mathcal{V}$, then $A \wedge B \in \mathcal{V}$.

The closed irreducible proofs of types $A \in \mathcal{V}$ have vector space properties.

Definition (Dimension of Proposition)

To each proposition $A \in \mathcal{V}$ we assign a dimension d inductively defined as:

- $d(\top) = 1$
- $d(A \wedge B) = d(A) + d(B)$.

Vector Space Properties 1

Definition (Zero Vector)

The zero vector 0_A of a type $A \in \mathcal{V}$ is inductively defined as:

- $0.\star$ if $A = \top$
- $\langle 0_B, 0_C \rangle$ if $A = B \wedge C$.

Definition (Additive Inverse)

The additive inverse $-t$ of closed irreducible proof $t : A$ of a type $A \in \mathcal{V}$ is inductively defined as:

- $(-a).\star$ if $A = \top$ and $t = a.\star$
- $\langle -t_B, -t_C \rangle$ if $A = B \wedge C$, $t = \langle t_B, t_C \rangle$, $t_B : B$ and $t_C : C$.

Vector Correspondence

Definition (One-to-One Correspondence)

Let $A \in \mathcal{V}$ be given and $n = d(A)$.
To each closed irreducible proof $t : A$ we assign a vector $\underline{t} \in \mathcal{S}^n$ inductively as follows:

- If $A = \top$ then $t = a.\star$. Choose $\underline{t} = \begin{pmatrix} a \end{pmatrix}$.
- If $A = B \wedge C$ then $t = \langle t_B, t_C \rangle$. Choose $\underline{t} = \begin{pmatrix} \underline{t_B} \\ \underline{t_C} \end{pmatrix}$.

To each vector $u \in \mathcal{S}^n$, assign a closed irreducible proof $\overline{u}^A : A$:

- If $n = 1$ then $u = \begin{pmatrix} a \end{pmatrix}$. Choose $\overline{u}^A = a.\star$.
- If $n > 1$ then $A = B \wedge C$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where u_1, u_2 are blocks of size $d(B), d(C)$ respectively. Choose $\overline{u}^A = \langle \overline{u}_1{}^B, \overline{u}_2{}^C \rangle$.

Example

Example

Let the vector $v = (1, 2, 3, 4)^T$,
$A = \top \wedge (\top \wedge (\top \wedge \top))$, and
$B = (\top \wedge \top) \wedge (\top \wedge \top)$ be given.
Then:

$$\overline{v}^A = \langle 1.\star, \langle 2.\star, \langle 3.\star, 4.\star \rangle\rangle\rangle,$$
$$\overline{v}^B = \langle\langle 1.\star, 2.\star\rangle, \langle 3.\star, 4.\star\rangle\rangle.$$

Vector Correspondence Properties

Lemma (Correspondence of sum)

For all $A \in \mathcal{V}$ and closed proofs $t, u : A$:

$$\underline{t \mathbin{\boldsymbol{+}} u} = \underline{t} + \underline{u}.$$

Lemma (Correspondence of scalar multiplication)

For all $A \in \mathcal{V}$, closed proof $t : A$ and $a \in \mathcal{S}$:

$$\underline{a \bullet t} = a \cdot \underline{t}.$$

Examples

Lemma (Correspondence of sum)

For all $A \in \mathcal{V}$ and closed proofs $t, u : A$:

$$\underline{t \boldsymbol{+} u} = \underline{t} + \underline{u}.$$

Example

Let $A = \top \wedge (\top \wedge \top)$, $t = \langle 1.\star, \langle 2.\star, 3.\star \rangle\rangle$, $u = \langle 4.\star, \langle 5.\star, 6.\star \rangle\rangle$. Then:

$$t \boldsymbol{+} u \to^* \langle 5.\star, \langle 7.\star, 9.\star \rangle\rangle$$

$$\underline{t \boldsymbol{+} u} = (5, 7, 9)^T$$

$$\underline{t} = (1, 2, 3)^T$$

$$\underline{u} = (4, 5, 6)^T$$

$$\underline{t} + \underline{u} = (5, 7, 9)^T$$

Examples

Lemma (Correspondence of scalar multiplication)

For all $A \in \mathcal{V}$, closed proof $t : A$ and $a \in \mathcal{S}$:

$$\underline{a \bullet t} = a \cdot \underline{t}.$$

Example

Let $A = \top \wedge (\top \wedge \top)$, $t = \langle 1.\star, \langle 2.\star, 3.\star \rangle\rangle$, $a = 2$. Then:

$$\begin{aligned} a \bullet t &\to \langle 2 \bullet 1.\star, 2 \bullet \langle 2.\star, 3.\star \rangle\rangle \\ &\to^* \langle 2.\star, \langle 4.\star, 6.\star \rangle. \\ \underline{a \bullet t} &= (2, 4, 6)^T \\ \underline{t} &= (1, 2, 3)^T \\ 2 \cdot \underline{t} &= (2, 4, 6)^T \end{aligned}$$

Vector Properties

Lemma

Let $A \in \mathcal{V}$, $a, b \in \mathcal{S}$, and closed proofs $t, t_1, t_2, t_3 : A$ be given. Then:

1. $(t_1 \mathbin{\textbf{+}} t_2) \mathbin{\textbf{+}} t_3 \equiv t_1 \mathbin{\textbf{+}} (t_2 \mathbin{\textbf{+}} t_3)$
2. $t_1 \mathbin{\textbf{+}} t_2 \equiv t_2 \mathbin{\textbf{+}} t_1$
3. $t \mathbin{\textbf{+}} 0_A \equiv t$
4. $t \mathbin{\textbf{+}} -t \equiv 0_A$
5. $a \bullet b \bullet t \equiv (a \wedge b) \bullet t$
6. $1 \bullet t \equiv t$
7. $a \bullet (t_1 \mathbin{\textbf{+}} t_2) \equiv a \bullet t_1 \mathbin{\textbf{+}} a \bullet t_2$
8. $(a + b) \bullet t \equiv a \bullet t \mathbin{\textbf{+}} b \bullet t$.

Matrices

Theorem (Matrix Correspondence Theorem)

Let $A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$, and let $M \in \mathcal{S}^{n,m}$ be a matrix with n rows and m columns. Then there is a closed proof $t : A \Rightarrow B$, such that for all $u \in \mathcal{S}^m$, $\underline{t\, \overline{u}^A} = Mu$.

Matrices

Theorem (Matrix Correspondence Theorem)

Let $A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$, and let $M \in \mathcal{S}^{n,m}$ be a matrix with n rows and m columns. Then there is a closed proof $t : A \Rightarrow B$, such that for all $u \in \mathcal{S}^m$, $\underline{t\,\overline{u}^A} = Mu$.

Definition

Inductively define the proof t_M for the matrix M as follows:

- If $A = \top$, then M has 1 column, thus has a corresponding vector $\overline{M}^B$. Take $t_M = \lambda x.\delta_\top(x, \overline{M}^B)$.
- If $A = A_1 \wedge A_2$, then $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix}$, where M_1 has $d(A_1)$ and M_2 has $d(A_2)$ columns. Take $t_M = \lambda x.(\delta^1_\wedge(x, z.t_{M_1}\, z) + \delta^2_\wedge(x, z.t_{M_2}\, z))$.

Example Matrix

Example

Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$.

Give the term $t : A \Rightarrow B$ corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$.

$$t = \lambda x.?$$

Example Matrix

Example

Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$.

Give the term $t : A \Rightarrow B$ corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$.

$$t = \lambda x.(\delta^1_\wedge(x, z.?)) \\ +(\delta^2_\wedge(x, y.?))$$

Example Matrix

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Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$.

Give the term $t : A \Rightarrow B$ corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$.

$$
\begin{aligned}
t = \lambda x.(\delta^1_\wedge(x, z.?))& \\
+ (\delta^2_\wedge(x, y.(\delta^1_\wedge(y, z.?))& \\
+(\delta^2_\wedge(y, z.?))))&
\end{aligned}
$$

Example Matrix

Example

Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$.

Give the term $t : A \Rightarrow B$ corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$.

$$\begin{aligned} t = \lambda x.(\delta^1_\wedge(x, z.\delta_\top(z, \langle 1.\star, 2.\star \rangle))) \\ + (\delta^2_\wedge(x, y.(\delta^1_\wedge(y, z.\delta_\top(z, \langle 2.\star, 3.\star \rangle)) \\ + (\delta^2_\wedge(y, z.\delta_\top(z, \langle 3.\star, 4.\star \rangle))))) \end{aligned}$$

Example Matrix Correspondence

Theorem (Matrix Correspondence)

Let $A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$, and let $M \in \mathcal{S}^{n,m}$ be a matrix with n rows and m columns. Then there is a closed proof $t : A \Rightarrow B$, such that for all $u \in \mathcal{S}^m$, $\underline{t\,\overline{u}^A} = Mu$.

Example

Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$. Choose $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$. Then:

$$\begin{aligned} t = \lambda x.(&\delta^1_\wedge(x, z.\delta_\top(z, \langle 1.\star, 2.\star\rangle))) \\ &+ (\delta^2_\wedge(x, y.(\delta^1_\wedge(y, z.\delta_\top(z, \langle 2.\star, 3.\star\rangle)) \\ &+ (\delta^2_\wedge(y, z.\delta_\top(z, \langle 3.\star, 4.\star\rangle))))) \end{aligned}$$

Choose $u = (3, 2, 1)^T$, hence: $\overline{u}^A = \langle 3.\star, \langle 2.\star, 1.\star\rangle\rangle$.

Example Matrix Correspondence

Theorem (Matrix Correspondence)

Let $A, B \in \mathcal{V}$ with $d(A) = m$ and $d(B) = n$, and let $M \in \mathcal{S}^{n,m}$ be a matrix with n rows and m columns. Then there is a closed proof $t : A \Rightarrow B$, such that for all $u \in \mathcal{S}^m$, $\underline{t\,\overline{u}^A} = Mu$.

Example

Let $A = \top \wedge (\top \wedge \top)$ and $B = \top \wedge \top$.

Choose $M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $u = \begin{pmatrix} 3, 2, 1 \end{pmatrix}^T$.

Then:

$$M\,u = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 16 \end{pmatrix}.$$

Example Matrix Correspondence

Example

Apply the term $t : A \Rightarrow B$ to the vector $\overline{u}^A = \langle 3.\star, \langle 2.\star, 1.\star\rangle\rangle$:

$$\begin{aligned} t = \lambda x.(\delta^1_\wedge(x, z.\delta_\top(z, \langle 1.\star, 2.\star\rangle))) \\ + (\delta^2_\wedge(x, y.(\delta^1_\wedge(y, z.\delta_\top(z, \langle 2.\star, 3.\star\rangle)) \\ + (\delta^2_\wedge(y, z.\delta_\top(z, \langle 3.\star, 4.\star\rangle))))) \end{aligned}$$

$$\begin{aligned} t\,\overline{u}^A &\to^* (\delta_\top(3.\star, \langle 1.\star, 2.\star\rangle))) \\ &\quad + (\delta_\top(2.\star, \langle 2.\star, 3.\star\rangle)) \\ &\quad + (1.\star, \langle 3.\star, 4.\star\rangle)) \\ &\to^* 3 \bullet \langle 1.\star, 2.\star\rangle + 2 \bullet \langle 2.\star, 3.\star\rangle + 1 \bullet \langle 3.\star, 4.\star\rangle \\ &\to^* \langle 10.\star, 16.\star\rangle. \end{aligned}$$

Hence

$$\underline{t\,\overline{u}^A} = \begin{pmatrix} 10 \\ 16 \end{pmatrix} = Mu.$$

Outline

Linearity of Proofs

Theorem (Closed proofs are linear)

If $A, B \in \mathcal{V}$, then each closed proof $t : A \Rightarrow B$ is linear, that is:

$$t\,(u + v) = t\,u + t\,v \quad \textit{and} \quad t\,(a \bullet u) = a \bullet t\,u.$$

This theorem follows from the following more general lemma:

Lemma (Linearity)

For every proposition A, proposition $B \in \mathcal{V}$, proofs t, u_1 and u_2, such that $x : A \vdash t : B$, t is irreducible, $\vdash u_1 : A$, and $\vdash u_2 : A$. Then

$$t\{u_1 + u_2\} \equiv t\{u_1\} + t\{u_2\} \qquad \textit{and} \qquad t\{a \bullet u_1\} \equiv a \bullet t\{u_1\}.$$

Definition (Substitution of single variable)

Given a term t, if $FV(t) \subseteq \{x\}$ for some variable x, then $t\{u\} := t[x := u]$.

Linearity of Proofs

Lemma (Linearity)

For every proposition A, proposition $B \in \mathcal{V}$, proofs t, u_1 and u_2, such that $x : A \vdash t : B$, t is irreducible, $\vdash u_1 : A$, and $\vdash u_2 : A$. Then

$$t\{u_1 \mathbin{\mathbf{+}} u_2\} \equiv t\{u_1\} \mathbin{\mathbf{+}} t\{u_2\} \qquad \text{and} \qquad t\{a \bullet u_1\} \equiv a \bullet t\{u_1\}.$$

Idea of proof

- The proof is done by induction on the size of a proof, using a size function μ.
- For the elimination case, define the notion of an elimination context K, similar to that of a head variable t of $t\, u_1\, u_2\, u_3\, \ldots$.
- The elimination context is used to decompose a proof t into $K\{u\}$, with $\mu(K) < \mu(t)$ and $\mu(u) < \mu(t)$, so the induction hypothesis can be applied twice.

Example

Example (Linear λ-term)

Consider the following term t from $\top \wedge \top \to \top$:

$$t = \lambda x.\delta^1_\wedge(x, y.y) \boldsymbol{+} 2 \bullet \delta^2_\wedge(x, y.y))$$

Take $u_1 = \langle a_1.\star, b_1.\star\rangle$ and $u_2 = \langle a_2.\star, b_2.\star\rangle$. Then $u_1 \boldsymbol{+} u_2 \equiv \langle (a_1 + a_2).\star, (b_1 + b_2).\star\rangle$.

Example

Example (Linear λ-term)

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Take $u_1 = \langle a_1.\star, b_1.\star\rangle$ and $u_2 = \langle a_2.\star, b_2.\star\rangle$. Then
$u_1 \boldsymbol{+} u_2 \equiv \langle (a_1 + a_2).\star, (b_1 + b_2).\star\rangle$.
By reduction we find:

$$\begin{aligned} t(\langle a, b\rangle) &= \delta^1_\wedge(\langle a.\star, b.\star\rangle, y.y) \boldsymbol{+} 2 \bullet \delta^2_\wedge(\langle a.\star, b.\star\rangle, y.y) \\ &\to^* a. \star \boldsymbol{+} 2 \cdot (b.\star) \\ &\to^* (a + 2 \cdot b).\star \end{aligned}$$

Example

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Take $u_1 = \langle a_1.\star, b_1.\star\rangle$ and $u_2 = \langle a_2.\star, b_2.\star\rangle$. Then
$u_1 \mathbin{\boldsymbol{+}} u_2 \equiv \langle (a_1 + a_2).\star, (b_1 + b_2).\star\rangle$.
By reduction we find: $t(\langle a, b\rangle) \to^* (a + 2 \cdot b).\star$.
Hence:

$$\begin{aligned} t\,(u_1 \mathbin{\boldsymbol{+}} u_2) &\to^* ((a_1 + a_2) + 2 * (b_1 + b_2)).\star \\ &= (a_1 + a_2 + 2 \cdot b_1 + 2 \cdot b_2).\star \\ &\leftarrow (a_1 + 2 \cdot b_1).\star \mathbin{\boldsymbol{+}} (a_2 + 2 \cdot b_2).\star \mathbin{\boldsymbol{+}} \\ &\leftarrow^* t\,u_1 \mathbin{\boldsymbol{+}} t\,u_2 \end{aligned}$$

Example

Example (Linear λ-term)

Consider the following term t from $\top \wedge \top \to \top$:

$$t = \lambda x.\delta^1_\wedge(x, y.y) \mathbf{+} 2 \bullet \delta^2_\wedge(x, y.y))$$

Take $u = \langle a.\star, b.\star \rangle$ and $c \in \mathcal{S}$. Then $c \bullet u \equiv \langle ca.\star, cb.\star \rangle$
Already determined that $t\,u \to^* (2a + b).\star$.

Example

Example (Linear λ-term)

Consider the following term t from $\top \wedge \top \to \top$:

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Already determined that $t\,u \to^* (2a + b).\star$.
By reduction we find:

$$\begin{aligned} t(c \bullet u) &\to t(\langle ca.\star, cb.\star \rangle) \\ &\to^* (2ca + cb).\star\,. \end{aligned}$$

Example

Example (Linear λ-term)

Consider the following term t from $\top \wedge \top \to \top$:

$$t = \lambda x.\delta^1_\wedge(x, y.y) \mathbf{+}\, 2 \bullet \delta^2_\wedge(x, y.y))$$

Take $u = \langle a.\star, b.\star\rangle$ and $c \in \mathcal{S}$. Then $c \bullet u \equiv \langle ca.\star, cb.\star\rangle$
Already determined that $t\,u \to^* (2a + b).\star$.
By reduction we find:

$$\begin{aligned} t(c \bullet u) &\to t(\langle ca.\star, cb.\star\rangle) \\ &\to^* (2ca + cb).\star\,. \end{aligned}$$

Finally, we determine that:

$$\begin{aligned} c \bullet (t\,u) &\to^* c \bullet ((2a + b).\star) \\ &\to (c \cdot (2a + b)).\star = (2ca + cb).\star\,. \end{aligned}$$

Example

Example (Non-linear λ-term)

Consider the following term t of type $\top \to (\top \to \top)$:

$$t = \lambda x.\lambda y.\delta_\top(x, y).$$

Then:

$$t\,(1.\star \mathbf{+} 2.\star) \to^* \lambda y.3 \bullet y$$
$$(t\,1.\star) \mathbf{+} (t\,2.\star) \to^* (\lambda y.1 \bullet y \mathbf{+} 2 \bullet y),$$

which are both irreducible but unequal.

Example

Example (Non-linear λ-term)

Consider the following term t of type $\top \to (\top \to \top)$:

$$t = \lambda x.\lambda y.\delta_\top(x, y).$$

Then:

$$t\,(1.\star + 2.\star) \to^* \lambda y.3 \bullet y$$
$$(t\,1.\star) + (t\,2.\star) \to^* (\lambda y.1 \bullet y + 2 \bullet y),$$

which are both irreducible but unequal.
However, applying the term $s = \lambda x.x\,(1.\star)$ to either result, we find that:

$$s\,(t\,(1.\star + 2.\star)) \to^* 3 \bullet 1.\star \to 3.\star$$
$$s\,(t\,1.\star) + (t\,2.\star) \to^* 1 \bullet 1.\star + 2 \bullet 1.\star \to^* 3.\star,$$

which are equivalent.

Remark

Remark

Whilst linearity does not generalize when $B \notin \mathcal{V}$, the results are still observationally equivalent in the following sense:
Let A, B be any type, let $t : A \Rightarrow B$ and let $C \in \mathcal{V}$. Then for all $s : B \Rightarrow C$:

$$s\,(t\,(u_1 + u_2)) \equiv s\,(t\,u_1 + t\,u_2) \quad \text{and} \quad s\,(t\,(a \bullet u_1)) \equiv s\,(a \bullet t\,u_1).$$

Outline

Quantum Programming

- Quantum programming languages are linear, except for the measurement operator.
- The Linear lambda calculus can be extended with non-deterministic pair $[,]$, $\delta^1_\odot$ and $\delta^2_\odot$ and $\delta_\odot$.
- Other than $\delta_\odot$, the non-deterministic pair is the same as conjunction, so all proofs excluding $\delta_\odot$ are linear.
- By introducing the single non-linear non-deterministic operator $\delta_\odot$, quantum algorithms can be represented.

Outline

Related Work: The algebraic lambda calculus

The Algebraic Lambda Calculus is an (untyped) Lambda Calculus.

Introduces addition of terms and multiplication of terms with a scalar in a Rig.

Defines an algebraic equality on terms to enforce linearity, rather than proving linearity.

Rig

Definition (Rig)

$R = (R, +, 0, \times, 1)$ is a rig (or semiring) if:

- R is a set of elements,
- $+, \times$ are binary operations,
- $(R, +, 0)$ is a commutative monoid,
- $(R, \times, 1)$ is a monoid,
- $\times$ is distrivutive over $+$ and
- 0 is absorbing over $\times$.

Definition (Positive Rig)

A rig R is called positive if for all $a, b \in R$, we have $a + b = 0$ implies $a = b = 0$.

Related Work: Properties of the algebraic lambda calculus

Proposition

If there is a negative element in the rig R, then all terms are equal.

Proof.

Let s and t be two terms, and Θ be the fixed point operator such that $\Theta\, f \to^* f(\Theta\, f)$. Define $\infty_s = \Theta\,(\lambda x.s + x)$. Then $\infty_s \equiv s + \infty_s$. Assume $1, -1 \in R$. Then:

$$
\begin{aligned}
s &\equiv s + (1-1)\infty_s + (1-1)\infty_t \\
&\equiv s + \infty_s - \infty_s + \infty_t - \infty_t \\
&\equiv t + \infty_s - \infty_s + \infty_t - \infty_t \\
&\equiv t
\end{aligned}
$$

□

Related Work: Properties of the algebraic lambda calculus

Definition (Finitely Splitting)

A rig R is called finitely splitting if for all $a \in R$, the set:

$$\{(a_1, \ldots, a_n) \in (R \setminus \{0\})^n \mid n \in N, a_1 + \ldots + a_n = a\},$$

is finite. A rig is called infinitely splitting if it is not finitely splitting.

Related Work: Properties of the algebraic lambda calculus

Definition (Finitely Splitting)

A rig R is called finitely splitting if for all $a \in R$, the set:

$$\{(a_1, \ldots, a_n) \in (R \setminus \{0\})^n \mid n \in N, a_1 + \ldots + a_n = a\},$$

is finite. A rig is called infinitely splitting if it is not finitely splitting.

Proposition

If an element in the Rig is infinitely splitting, then the Algebraic Lambda Calculus is not strongly normalizing for all terms.

Example

Suppose that the Rig is $\mathbb{Q}$, and s, s' are terms such that $s \to s'$. Then:

$$s \to \frac{1}{2}s + \frac{1}{2}s' \to \frac{1}{4}s + \frac{3}{4}s' \to \ldots$$

Related Work: Properties of the algebraic lambda calculus

Definition

The canonical form of a term σ is given by $can(\sigma)$ and is a unique representative in the algebraic equality equivalence class. In the paper an inductive definition of the canonical forms is given.

Definition

A term σ is called weakly typeable, denoted $\Gamma \vdash_R \sigma : A$, if the canonical form is typeable: $\Gamma \vdash_R can(\sigma) : A$.

Proposition

All weakly typeable terms in the algebraic lambda calculus are strongly normalizing.

Appendix

- Size of a proof
- Elimination contexts
- Sketch of linearity proof

Size of a proof

Definition (Size of a proof)

$$\mu(x) = 0$$
$$\mu(a.\star) = 1$$
$$\mu(a \bullet t) = \mu(\delta_\perp(t)) = \mu(\lambda x.t) = 1 + \mu(t)$$
$$\mu(inl(t)) = \mu(inr(t)) = 1 + \mu(t)$$
$$\mu(\delta_\top(t, u)) = \mu(t\, u) = 1 + \mu(t) + \mu(u)$$
$$\mu(\delta^1_\wedge(t, y.u)) = \mu(\delta^2_\wedge(t, y.u)) = 1 + \mu(t) + \mu(u)$$
$$\mu(t \boldsymbol{+} u) = 1 + \max(\mu(t), \mu(u))$$
$$\mu(\langle t, u\rangle) = 1 + \max(\mu(t), \mu(u))$$
$$\mu(\delta_\vee(t, y.u, z.v)) = 1 + \mu(t) + \max(\mu(u), \mu v)$$

This function was chosen with the following property:

$$\mu(t[x := u]) \leq \mu(t) + \mu(v).$$

Properties of size function

Lemma (Substitution Lemma)

If $\Gamma, x : A \vdash t : B$ *and* $\Delta \vdash u : A$*, then* $\mu(t[x := u]) \leq \mu(t) + \mu(u)$*.*

Equality would be possible if each closed proof had exactly one occurrence of x. However $\delta_\bot$ may not use the variable x.

Lemma (Subject Reduction)

If $t \to u$*, then* $\mu(t) \geq \mu(u)$*.*

Elimination Context

Definition (Elimination Context)

An elimination context is a proof with a single free variable _, in the language:

$$K = \_|\delta_{\top}(K, u)|\delta_{\bot}(K)|Kt|\delta^1_{\wedge}(K, x.r)|\delta^2_{\wedge}(K, x.r)|\delta_{\vee}(K, x.r, x.s),$$

where u is closed, $FV(r) \subseteq \{x\}$ and $FV(s) \subseteq \{y\}$.

Notation

In the case where a proof t has a single free variable x, the notation $t\{u\}$ is used instead of $t[x := u]$

Lemma

$$\mu(K\{t\}) = \mu(K) + \mu(t).$$

Elimination Context Properties

Lemma (Decomposition of a Proof)

Let t be an irreducible proof with $x : C \vdash t : A$. Then there is an elimination context K, a type B and a proof u such that:

$$\_ : B \vdash K : A, \qquad x : A \vdash u : B, \qquad t = K\{u\},$$

where u is a variable, introduction, sum or product.

Lemma (Decomposition of an Elimination Context)

If K is an elimination context such that $\_ : A \vdash K : B$ and $K \neq \_$, then K has form $K_1\{K_2\}$, and:

- *if $A = \top$ then K_2 has form $\delta_\top(\_, t)$,*
- *if $A = \bot$ then K_2 has form $\delta_\bot(\_)$,*
- *if $A = B \Rightarrow C$ then K_2 has form $\_t$,*
- *if $A = B \wedge C$, then K_2 has form $\delta^1_\wedge(\_, x.t)$ or $\delta^2_\wedge(\_, x.t)$,*
- *if $A = B \vee C$, then K_2 has form $\delta^2_\vee(\_, x.t, y.s)$.*

Proof sketch

Lemma (Linearity)

For every proposition A, proposition $B \in \mathcal{V}$, proofs t, u_1 and u_2, such that $x : A \vdash t : B$, t is irreducible, $\vdash u_1 : A$, and $\vdash u_2 : A$. Then

$$t\{u_1 \mathbin{\boldsymbol{+}} u_2\} \equiv t\{u_1\} \mathbin{\boldsymbol{+}} t\{u_2\} \qquad \textit{and} \qquad t\{a \bullet u_1\} \equiv a \bullet t\{u_1\}.$$

Sketch of Proof.

The proof is an induction on the size of the proof μ.
As t is irreducible, t is a variable, sum, product, elimination or introduction.
- The variable, sum and product cases follow from $\boldsymbol{+}$ and $\bullet$ properties.
- In the introduction case, $B \in \mathcal{V}$, so t is a pair. This case follows from reduction rules for pairs.
- In the elimination case, deconstruct t into $t = K\{v\}$, where v is a variable, sum or product. □

Proof sketch

Lemma (Linearity)

For every proposition A*, proposition* $B \in \mathcal{V}$*, proofs* t*,* u_1 *and* u_2*, such that* $x : A \vdash t : B$*,* t *is irreducible,* $\vdash u_1 : A$*, and* $\vdash u_2 : A$*. Then*

$$t\{u_1 + u_2\} \equiv t\{u_1\} + t\{u_2\} \qquad \text{and} \qquad t\{a \bullet u_1\} \equiv a \bullet t\{u_1\}.$$

Sketch of Proof Continued.

In the elimination case, deconstruct t into $t = K\{v\}$, where v is a variable, sum or product.

- If v is a variable, decompose the context into $K = K_1\{K_2\}$, then consider each case of K_2.

- If v is a sum or product, the subcase follows from the induction hypothesis on K and the reduction rules. □

The end

Are there questions?