Objecs of categories as complex numbers

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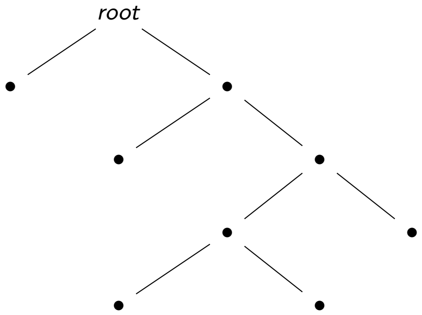
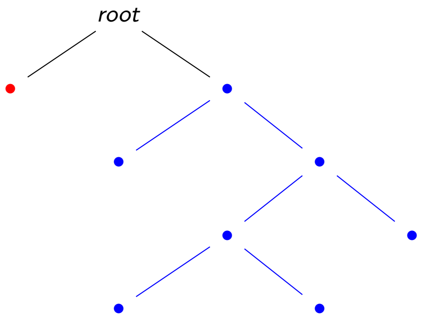
Outline

Trees



Trees



We say that the tree over here is $[t_1, t_2]$. Where t_1 is the left subtree, and t_2 is the right subtree, attached at the root.

Bijection between trees

Lawvrere

We have a **very specific** bijection between one tree T, and a seven tuple tree T^7.

Very specific/explicit

By a very specific bijection we mean a bijection that can be calculated in “constant time". I.e. the time is independent of the trees we have.

Bijection between trees

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We have a **very specific** bijection between one tree T, and a seven tuple tree T^7.

First prove

Let $t = (t_1, ..., t_7)$ be a tuple of seven trees, with at least one of trees 1,2,3,4 is non empty. Tree:

$$[[[[[[t_7, t_6]t_5]t_4]t_3]t_2]t_1]$$

If 1 upto 4 are empty then ... See Reference [4].

Bijection between trees

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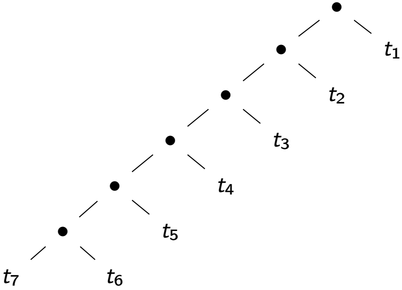

Another way of proving

One tree to two

We have a very explicit injection from one tree to a tuple of two trees.

$$T \to T^2$$

Proof.

We send tree t to the trees $(t, 0)$. □

Note

This is not a bijection and moreover there doesn't exists an explicit bijection.

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This is not a bijection and moreover there doesn't exists an explicit bijection.

$T \cong 1 + T^2$

However, we can define a very explicit bijection from T to $1 + T^2$ by attaching two trees with a root, or taking the empty tree.

From $T \cong 1 + T^2$ to $T^7 \cong T$

Complex numbers

Now we assume T is a complex number and we solve $T = 1 + T^2$ implying $T = e^{\frac{i\pi}{3}}$.

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Complex numbers

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Then $T^6 = 1$ but that is not an equivalence of trees...
However, we do have $T^7 = T$ as we have seen before

Proving with nuclear penny's

Let us prove the equivalence one more time, but now with a game.
First legal move:

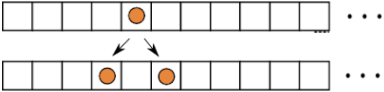

Figure: Fission

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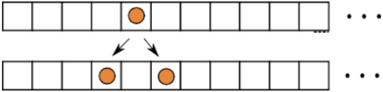


Figure: Fission

Second legal move:

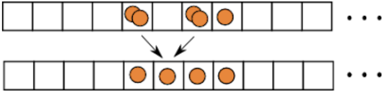


Figure: Fusion

See the blog by Sigfpe, Reference [1]

Question of nuclear penny's

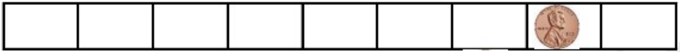
If this is the starting position:

Figure: Start

Can we reach this target position?

Figure: Target

Nuclear penny's game answer



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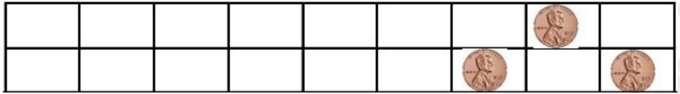

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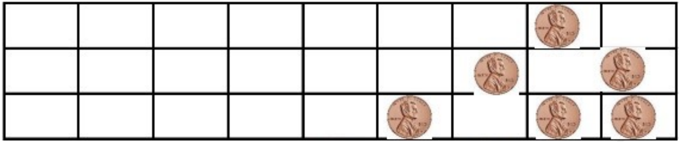

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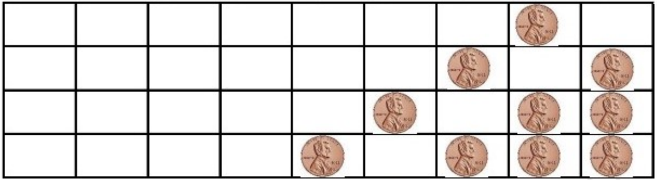

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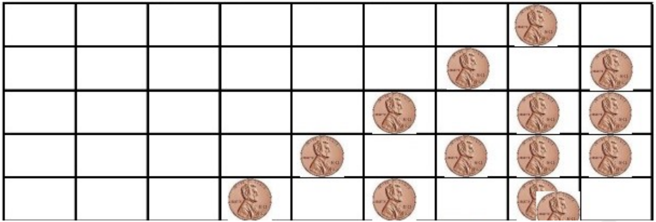

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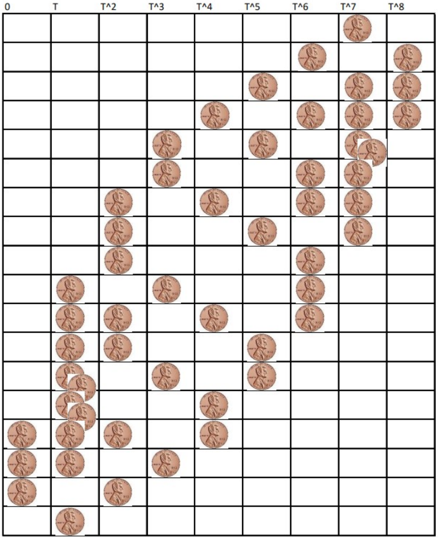

Outline

Goal

Recall

The fastest proof was to solve $T = 1 + T^2$ for complex numbers to find $T = T^7$.

Question

Can we use the complex numbers for more problems?

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The fastest proof was to solve $T = 1 + T^2$ for complex numbers to find $T = T^7$.

Question

Can we use the complex numbers for more problems?

Our main result is approximately

Let p, q_1 and q_2 be polynomials over $\mathbb{N}$: If

$$t = p(t) \implies q_1(t) = q_2(t)$$

for all complex numbers t, then

$$T \cong p(T) \implies q_1(T) \cong q_2(T)$$

for all objects T of any category in which it makes sense to add and multiply objects.

The way we are gonna prove

Distributive categories/ Haskell

$\Downarrow$ $\curvearrowleft$ burnside semiring

Semirings

$\Downarrow$ $\curvearrowleft$ catalysts

Ring

Outline

Distributive categories

Distributive category

A category $\mathscr{C}$ with finite products $\times$ and coproducts $\oplus$ is called distributive if, for all $A, B, C \in \mathscr{C}$ the following morphism is an isomorphism

$$k : A \times B \oplus A \times C \to A \times (B \oplus C) \tag{1}$$

In order to get this, we have two underlying morphisms

$$j : A \times B \to A \times (B \oplus C) \tag{2}$$

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Haskell

Let us have the objects A, B. We define the following operations:

- $A \times B$ as the Cartesian product or pair (A, B)
- $A \oplus B$ as `Either`$(A\ B)$

Haskell distributive

Lemma

Haskell is a distributive category

Proof.

We need to prove that we have an isomorphism

$$\texttt{Either}((A, B)\ (A, C)) \xrightarrow{\sim} (A, \texttt{Either}(B\ C)) \tag{4}$$

We can prove this, by showing that the two underlying morphisms are bijective.

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$$\texttt{left}[\texttt{Either}((A, B)\ (A, C))] = (A, B) = (A, \texttt{left}[\texttt{Either}(B\ C)])$$
$$\texttt{right}[\texttt{Either}(A, B)\ (A, C)] = (A, C) = (A, \texttt{right}[\texttt{Either}(B\ C)])$$

Note that these maps are bijective which concludes the argument. □

Recall rings

Semiring

A semiring is a set with operations $+$ and $\cdot$, such that for all a, b, c in the set:

- $(a+b)+c = a+(b+c)$ and $a+b = b+a$
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- There are neutral elements 0, 1 st $a+0 = a$ and $a \cdot 1 = a = 1 \cdot a$
- $a \cdot b + a \cdot c = a \cdot (b+c)$

Example: $(\mathbb{N}, +, \cdot)$

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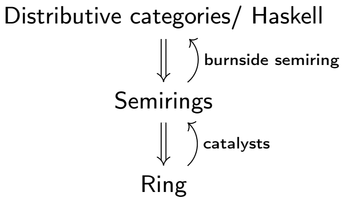
Ring

A ring is a semiring with the additional property that each element has an inverse under $+$, i.e. $a + -a = 0$.

Example: $(\mathbb{Z}, +, \cdot)$

The way we prove



Outline

Burnside semiring

Recall that in a distributive category we have the isomorphism

$$k : A \times B \oplus A \times C \xrightarrow{\sim} A \times (B \oplus C) \tag{5}$$

In semirings we have the equality

$$A \times B \oplus A \times C = A \times (B \oplus C) \tag{6}$$

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Burnside semiring

$$\text{Burnside semiring} \;=\; {}^{\text{distributive category}}\big/_{\text{isomorphisms}}$$

Semiring to Haskell

Distributive categories/ Haskell

$\Downarrow$ $\curvearrowleft$ burnside semiring

Semirings

Claim

Given a statement with *equality* in a semiring, we can relate it to a similar statement, but now with *isomorphism* in the distributive category.

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$\Downarrow$ $\uparrow$ burnside semiring

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Given a statement with *equality* in a semiring, we can relate it to a similar statement, but now with *isomorphism* in the distributive category.

Example

$$\forall \text{ semirings } A \text{ and all } a \in A; \text{ if } p_1(a) = p_2(a) \text{ then } q_1(a) = q_2(a)$$

Is equivalent with:

$$\forall \text{ distr. categories } \mathcal{A} \text{ and all } T \in \mathcal{A}; \text{ if } p_1(T) \cong p_2(T) \text{ then } q_1(T) \cong q_2(T).$$

The way we prove

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$\Downarrow$ $\curvearrowleft$ burnside semiring

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$\Downarrow$ $\curvearrowleft$ catalysts

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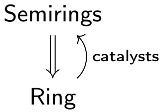
Outline

From rings to semirings

Note that the difference between rings and semirings is, that rings have subtraction.



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$$\begin{array}{c} \text{Semirings} \\ \Downarrow \quad \curvearrowleft \text{ catalysts} \\ \text{Ring} \end{array}$$

Proposition 7: Catalysts

Let $p_1; p_2; q_1; q_2 \in \mathbb{N}[x]$ and suppose that

$$p_1(x) = p_2(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically:}$$

Then there exists $s \in \mathbb{N}[x]$ such that

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a) For all rings A and all $a \in A$ if $p_1(a) = p_2(a)$ then $q_1(a) = q_2(a)$
b) as a), but restricted to commutative rings
c) q_1 and q_2 represent the same element of the quotient ring $\mathbb{Z}[x]/(p_1 - p_2)$
d) $p_1 - p_2$ divides $q_1 - q_2$ in the ring $\mathbb{Z}[x]$.

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Proof Proposition catalysts

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Proof.

We write $r = r_1 - r_2$ for some $r_1, r_2 \in \mathbb{N}[x]$, so $q_1 - q_2 = (r_1 - r_2)(p_1 - p_2)$ and then

$$q_1 + r_1 p_2 + r_2 p_1 = q_2 + r_1 p_1 + r_2 p_2 \text{ in } \mathbb{N}[x]$$

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set $s = r_1 p_1 + r_2 p_2$ in $\mathbb{N}[x]$, so $s = r_1 p_1 + r_2 p_1$ in $\mathbb{N}[x]/(p_1 = p_2)$ then:

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thus $q_1 + s$ and $q_2 + s$ represent the same element of the quotient semiring $\mathbb{N}[x]/(p_1 = p_2)$ as required. □

The way we prove

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Ring

Outline

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Theorem 17

First set of requirements. Second set of requirements. Then

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Proposition 2

Set of requirements implies:

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

Theorem 16

Set of requirements if:

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

then it also holds semring-theoretically.

Proof of Proposition 2

Proposition 2

Let $p, q_1, q_2 \in \mathbb{Z}[x]$ Suppose that the polynomial $p(x) - x \in \mathbb{Z}[x]$ is primitive (the coefficients are co-prime) and has no repeated complex roots, and that each complex root t satisfies $q_1(t) = q_2(t)$: Then

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

Proof.

This proof uses the division algorithm and Gauss lemma which can only be done in rings, not semirings. □

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Take $p(x) = 1 + x^2$ then we have $p(x) - x = 1 - x + x^2$. We see that $(1, -1, 1)$ are co-prime.

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$$(e^{\pm i\pi/3})^7 = e^{\pm i\pi/3}$$

Then now it should hold that $x = 1 + x^2 \Rightarrow x^7 = x$,

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$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

Example

Take $p(x) = 1 + x^2$ then we have $p(x) - x = 1 - x + x^2$. We see that $(1, -1, 1)$ are co-prime. The two complex roots are $e^{\pm i\pi/3}$, for both complex roots we have

$$(e^{\pm i\pi/3})^7 = e^{\pm i\pi/3}$$

Then now it should hold that $x = 1 + x^2 \Rightarrow x^7 = x$,so we should find an $r \in \mathbb{Z}[x]$ such that $x^7 - x = r(1 - x + x^2)$.

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$$\begin{aligned} x^7 - x &= (x^5 + x^4 - x^2 - x)(1 - x + x^2) \\ &= x^5 + x^4 - x^2 - x - x^6 - x^5 + x^3 + x^2 + x^7 + -x^6 - x^4 - x^3 \\ &= x^7 - x \end{aligned}$$

Catalysts

Recall Theorem 16

Let p, q_1, q_2 be polynomials such that p has non-zero constant term and degree at least two, and q_1 and q_2 have degree at least one. If

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

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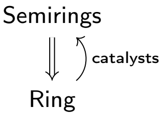

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Recall Proposition 7: Catalysts

Let $p_1; p_2; q_1; q_2 \in \mathbb{N}[x]$ and suppose that

$$p_1(x) = p_2(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically:}$$

Then there exists $s \in \mathbb{N}[x]$ such that

$$p_1(x) = p_2(x) \Rightarrow q_1(x) + s(x) = q_2(x) + s(x) \text{ semiring-theoretically:}$$

Outline

High elements

Definition

Given a set A with operation $*$, we say $b \leq a$

$$\exists c \in A \text{ such that } b * c = a$$

An element $a \in A$ is called High if **for all** $b \in A$ we have $b \leq a$. The High elements of A are denoted by $\mathrm{H}(A)$.

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Properties of High elements

- If a is High and $a \leq b$, then b is high.
- If a is High then for all b we have $a * b$ is high (since $a \leq a * b$).

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Example: all elements of $(\mathbb{Z}, +)$ are high, for all a, b there exists a c such that $b + c = a$ (take $c = a - b$).
Non example: $(\mathbb{N}, +)$ has no high element, since for all n we have that for $n + 1$ we cannot find a c such that $n + 1 + c = n$.

Group of High elements

Lemma

The High elements of a commutative semigroup A form a group.

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Take $d \in \mathrm{H}(A)$, since it is High $\exists z \in \mathrm{H}(A)$ such that $d * z = d$.

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The High elements of a commutative semigroup A form a group.

Proof.

Take $d \in \mathrm{H}(A)$, since it is High $\exists z \in \mathrm{H}(A)$ such that $d * z = d$.
Claim: z is the unit. This is because $\forall b \in \mathrm{H}(A)$ $\exists c$ such that

$$d * c = b$$
$$z * d * c = b$$
$$z * b = b.$$

So we conclude that z is indeed the unit.

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So we conclude that z is indeed the unit.
Also each element has an inverse, since $z \in \mathrm{H}(A)$ and so $\forall b \in \mathrm{H}(A) \exists c$ such that $b * c = z$, we say c is the inverse of b. □

High elements continued

Corollary 11

If a_1, a_2 are High elements of a commutative semi-group $(A, *)$ and if there exists an $b \in A$ such that $a_1 * b = a_2 * b$ then $a_1 = a_2$.

High elements continued

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Proof.

Take any High element $\tilde{a}$. Then we write

$$a_1 * b * \tilde{a} = a_2 * b * \tilde{a}$$
$$a_1 * (b * \tilde{a}) = a_2 * (b * \tilde{a})$$

Note that $b * \tilde{a}$ is an High element since $\tilde{a}$ is.

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Note that $b * \tilde{a}$ is an High element since $\tilde{a}$ is.
Now since the high elements form a group with inverses, there is an inverse of $b * \tilde{a}$ so we can conclude $a_1 = a_2$. □

High polynomials

Fix a polynomial $p \in (\mathbb{N}[x], \cdot)$ and form the quotient $\mathbb{N}[x]/(x = p(x))$.
We will write $f \leq g$ to say that $f \leq g \in \mathbb{N}[x]/(x = p(x))$

Lemma 13

If p has non-zero constant term then $1 \leq x \leq x^2 \leq \ldots$

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Proof.

By assumption $1 \leq p$, so $1 \leq x$,then iteratively multiply both sides with x to make the sequence $1 \leq x \leq x^2 \leq \ldots$ □

High polynomials continued

Lemma 14

If p has non-zero constant term and degree ≥ 2 then we have:

- $x \geq nx$
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We prove by induction. We show only the case $n = 2$, then we can multiply to see it holds for all n.
By assumption $p(x) \geq 1 + x^d$, for $d \geq 2$ so $x \geq 1 + x^d$.

$$x \geq x^d$$

For the second bullet, we see $x \geq x^d$ and since $d \geq 2$ we have $x \geq x^2$. □

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High polynomials continued 2

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Proposition 15

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Proof.

By lemma 13 we see that if we prove that x is high, then all others are to. By lemma 14 we see that x is high, so we are done. □

Outline

Proof of Theorem 16

Theorem 16

Let p, q_1, q_2 be polynomials such that p has non-zero constant term and degree at least two, and q_1 and q_2 have degree at least one. If

$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically}$$

then it also holds semring-theoretically.

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Note

Since q_1 and q_2 are non-constant, they are high.

Proof continued

Corollary 11

If a_1, a_2 are High elements of a commutative semi-group $(A, *)$ and if there exists an $b \in A$ such that $a_1 * b = a_2 * b$ then $a_1 = a_2$.

Note

Since q_1, q_2 are High elements of $(\mathbb{N}, +)$, if there is an $s \in \mathrm{N}[x]$ such that $q_1(x) + s(x) = q_2(x) + s(x)$ then $q_1 = q_2$

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Proposition 7: Catalysts

Let $p_1; p_2; q_1; q_2 \in \mathbb{N}[x]$ and suppose that

$$p_1(x) = p_2(x) \Rightarrow q_1(x) = q_2(x) \text{ ring-theoretically:}$$

Then there exists $s \in \mathbb{N}[x]$ such that

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then it also holds semring-theoretically.

Proof.

Since q_1 and q_2 are high, and there exists a catalyst such that $q_1(x) + s(x) = q_2(x) + s(x)$ holds semiring theoretically, we conclude that $q_1 = q_2$ holds semiring theoretically. □

Recall

Proposition 2

Let $p, q_1, q_2 \in \mathbb{Z}[x]$ Suppose that the polynomial $p(x) - x \ \in \mathbb{Z}[x]$ is primitive and has no repeated complex roots, and that each complex root t satisfies $q_1(t) = q_2(t)$: Then

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Goal

Theorem 17

Let p, q_1, q_2 be polynomials such that p has non-zero constant term and degree at least two, and q_1 and q_2 have degree at least one. Suppose that the polynomial $p(x) - x \in \mathbb{Z}[x]$ is primitive and has no repeated complex roots, and that each complex root t satisfies $q_1(t) = q_2(t)$: Then

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Distributive categories/ Haskell

$\Downarrow$ burnside semiring

Semirings

Goal

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$$x = p(x) \Rightarrow q_1(x) = q_2(x) \text{ semiring-theoretically}$$

Use of the burnside semiring

$$\forall \text{ semirings } A \text{ and all } a \in A; \text{ if } p_1(a) = p_2(a) \text{ then } q_1(a) = q_2(a)$$

Is equivalent with:

$$\forall \text{ distr. categories } \mathcal{A} \text{ and all } T \in \mathcal{A}; \text{ if } p_1(T) \cong p_2(T) \text{ then } q_1(T) \cong q_2(T).$$

Example

So if $c = c^2 + 1$ implies $c = c^7$ for all complex number c, then $T \cong T^2 + 1$ implies $T \cong T^7$ for all trees T.

Outline

Gaussian integers

Definition

Gaussian integers are complex numbers $z = ai + b$ for which $a, b \in \mathbb{Z}$.

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$\mathbb{N}[x]/(x = 1 + x + x^2)$ is isomorphic to the ring of Gaussian integers ($\mathscr{R}$).

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Theorem

$\mathbb{N}[x]/(x = 1 + x + x^2)$ is isomorphic to the ring of Gaussian integers ($\mathscr{R}$).

Proof.

We see

$$\mathbb{N}[x]/(x = 1 + x + x^2) \cong \mathbb{Z}[x]/(1 + x + x^2 - x) = \mathbb{Z}[x]/(1 + x^2)$$

We see the ring of Gaussian integers ($\mathscr{R}$) consists of elements $m + ni$ with $m, n \in \mathbb{Z}$. If we take the mapping $m + nx^2 \Leftrightarrow m + ni$ we see $\mathbb{Z}[x]/(1 + x^2) \cong \mathscr{R}$. □

Gaussian ring

Note

Note moreover that if $x = i$ we have $\mathcal{R} \cong \mathbb{Z}[i]/(1 + i^2) \cong \mathbb{Z}[i]$.

Gaussian integers applied

Result from before

If $x = 1 + x + x^2$ implies $x = x^5$ as before, then

$$\forall \text{ distr. categories } \mathscr{A} \text{ and all } T \in \mathscr{A}; \text{ if } T \cong 1 + T + T^2 \text{ then } T \cong T^5.$$

Gaussian integers applied

Result from before

If $x = 1 + x + x^2$ implies $x = x^5$ as before, then

$$\forall \text{ distr. categories } \mathcal{A} \text{ and all } T \in \mathcal{A}; \text{ if } T \cong 1 + T + T^2 \text{ then } T \cong T^5.$$

Note

Take $p(x) = 1 + x + x^2$. Then $x = 1 + x + x^2$ implies $x = \pm i$. So we have $x = x^5$.
Then we can conclude that if $T \cong 1 + T + T^2$ then $T \cong T^5$.

Motzkin trees

Definition

A Motzkin tree is tree where every node has either 0 1 or 2 children.

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For Motzkin trees we have $T \cong 1 + T + T^2$.

Proof.

Take a node in a tree there are 3 possibilities:

- Either the node has 0 children, this corresponds to 1.
- Or it has one child, there is a tree attached to it, this corresponds to T.
- Or it has two children, both with a tree attached, this corresponds to T^2.

□

Conclusion

Motzkin trees

Before we had $x = 1 + x + x^2$ implies $x = x^5$.
Since we have proven that for Motzkin trees $T \cong 1 + T + T^2$, we can conclude that $T \cong T^5$ for all Motzkin trees.

Gaussian penny's

Recall we have $x = 1 + x + x^2$ First legal move:

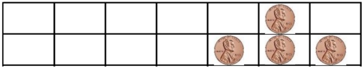

Figure: Fission

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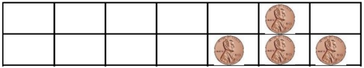


Figure: Fission

Second legal move:

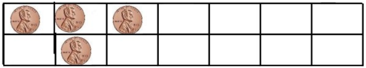


Figure: Fusion

Question of nuclear penny's

If this is the starting position:

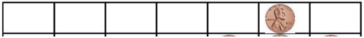


Figure: Start

Can we reach this target position?

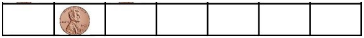
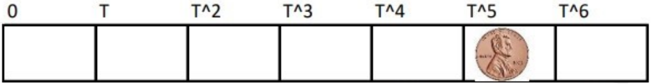


Figure: Target

Nuclear penny's game answer



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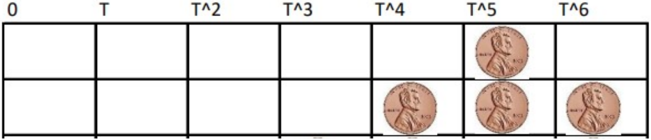

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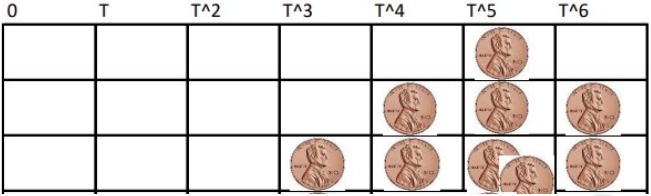

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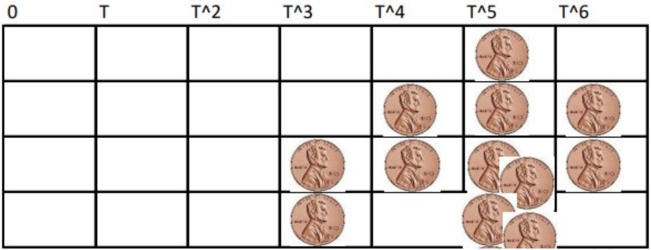

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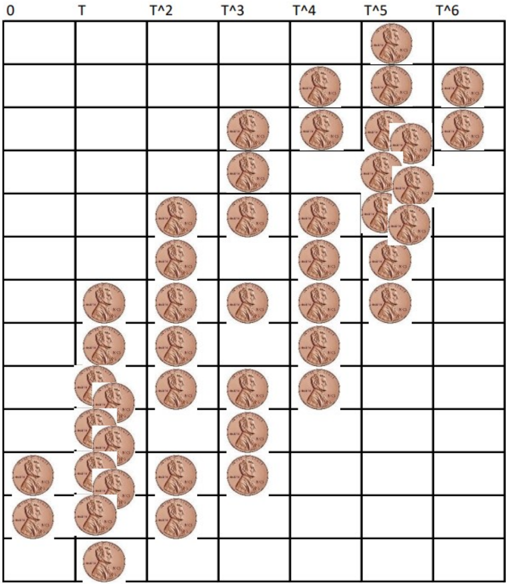

Outline

Reference

Reference

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Questions

Are there any questions?