Completeness Theorems for Kleene Algebra with Top

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Part I: Kleene Algebra[1]

Syntax

Definition (Syntax of regular expressions)

Given a set of symbols $\Sigma = \{a, b, c, \dots\}$, *regular expressions* are inductively defined with

$$e, f ::= e + f \mid e \cdot f \mid e^* \mid 0 \mid 1 \mid a \in \Sigma$$

Examples

$$(a + b) \cdot c^*$$

$$a \cdot (b + c \cdot d)$$

Semantics

Interpretation (or meaning) of a regular expression

Example ($(a + b) \cdot c^*$)

The set of accepted words is $\{\mathtt{a}, \mathtt{b}, \mathtt{ac}, \mathtt{bc}, \mathtt{acc}, \mathtt{bcc}, \dots\}$
Thus $[\![(a + b) \cdot c^*]\!] = \{\mathtt{a}, \mathtt{b}, \mathtt{ac}, \mathtt{bc}, \dots\}$

Definition (Semantics for regular expressions)

$$[\![0]\!] = \emptyset \qquad [\![1]\!] = \{\epsilon\} \qquad [\![a]\!] = \{\mathtt{a}\}$$

$$[\![e + f]\!] = [\![e]\!] \cup [\![f]\!] \qquad [\![e \cdot f]\!] = [\![e]\!] \cdot [\![f]\!] \qquad [\![e^*]\!] = [\![e]\!]^*$$

Kleene Algebra

- Set of axioms like $e + e = e$
- Does $(a + b) \cdot c^* = a \cdot c^* + b \cdot (c \cdot c^*) + b$ hold?

Definition (Kleene algebra)

We define $=$ as the smallest *congruence* on all regular expressions e, f, g, satisfying the following rules:

$e + (f + g) = (e + f) + g$

$e^* = 1 + e \cdot e^* \qquad e^* = 1 + e^* \cdot e$

$\ldots$

- We write $\mathbf{KA} \vdash e = f$, if $e = f$ can be derived with the Kleene algebra axioms using equational logic
  $\Rightarrow$ e and f are provably equivalent

Soundness and Completeness

Theorem (Soundness)

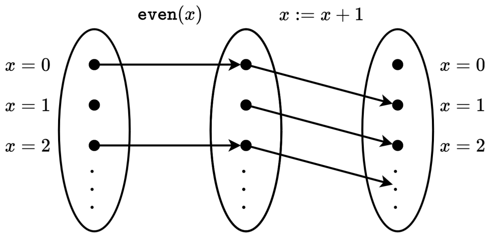
$$\mathbf{KA} \vdash e = f \;\Rightarrow\; [\![e]\!] = [\![f]\!]$$

Theorem (Completeness)

$$[\![e]\!] = [\![f]\!] \;\Rightarrow\; \mathbf{KA} \vdash e = f$$

Relational Model

- Think of $a \in \Sigma$ as a binary relation on $\mathbb{N}$
- Think of regular expressions as programs
- $\text{even}(x) = \{\langle x, x\rangle \mid x \in \mathbb{N}, x \text{ is even}\}$
- $x := x + 1 \quad = \quad \{\langle x, x+1\rangle \mid x \in \mathbb{N}\}$



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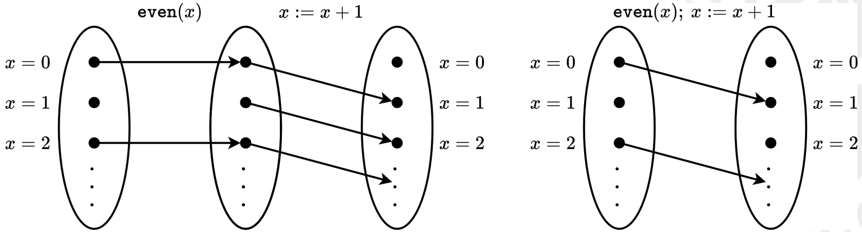
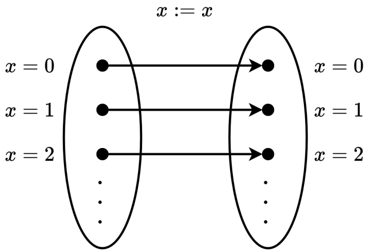
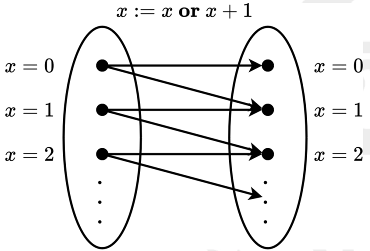


Relational Model examples





Relational Model (formal)

- A relational interpretation of $\Sigma = \{a, b, \dots\}$ consists of
  - a set X
  - a function $\sigma : \Sigma \to \mathcal{P}(X \times X)$
- $\mathbf{REL} \vDash e = f$

Definition (Semantics of the relational model)

$$[\![0]\!]_\mathsf{R} = \emptyset \qquad [\![1]\!]_\mathsf{R} = \{\langle x, x\rangle \mid x \in X\} \qquad [\![a]\!]_\mathsf{R} = \sigma(a)$$

$$[\![e + f]\!]_\mathsf{R} = [\![e]\!]_\mathsf{R} \cup [\![f]\!]_\mathsf{R} \qquad [\![e^*]\!]_\mathsf{R} = [\![1]\!]_\mathsf{R} \cup [\![e]\!]_\mathsf{R} \cup [\![e \cdot e]\!]_\mathsf{R} \cup \dots$$

$$[\![e \cdot f]\!]_\mathsf{R} = \{\langle x, z\rangle : \exists y \in X \,.\, \langle x, y\rangle \in [\![e]\!]_\mathsf{R} \wedge \langle y, z\rangle \in [\![f]\!]_\mathsf{R}\}$$

Relational Model examples (cont.)

- $i := 0;$
  $\texttt{while } (i+1)^2 \leq n \texttt{ do}$
  $\quad | \quad i := i + 1;$
- $\Sigma = \{a, b, c, d\}$
- $X = \mathbb{N} \times \mathbb{N}$
- $\sigma(a) = \{ \langle (n, i), (n, 0) \rangle \mid n, i \in \mathbb{N}\}$
  $\sigma(b) = \{ \langle (n, i), (n, i) \rangle \mid n, i \in \mathbb{N}, (i+1)^2 \leq n\}$
  $\sigma(c) = \{ \langle (n, i), (n, i+1) \rangle \mid n, i \in \mathbb{N}\}$
  $\sigma(d) = \{ \langle (n, i), (n, i) \rangle \mid n, i \in \mathbb{N}, (i+1)^2 > n\}$
- $[\![a \cdot (b \cdot c)^* \cdot d]\!]_{\mathrm{R}} = \{ \langle (n, i), (n, \mathrm{IntSqrt}(n)) \rangle \mid n, i \in \mathbb{N}\}$

Summary of part I

- Regular expressions: $e, f ::= e + f \mid e \cdot f \mid e^* \mid 0 \mid 1 \mid a \in \Sigma$
- Kleene algebra: set of equations between regular expressions
- Language model: $[\![e]\!]_{\mathsf{L}} \subseteq \Sigma^*$
- Relational model: Set X and $\sigma : \Sigma \to \mathcal{P}(X \times X)$
  $[\![e]\!]_{\mathsf{R}} \subseteq X \times X$
- **KA** is sound and complete w.r.t. **LANG** and **REL**

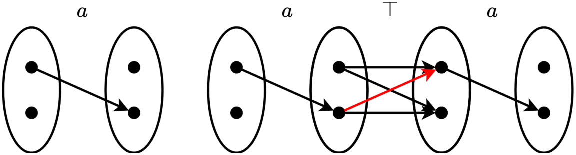
Part II: Kleene Algebra with Top[2]

What is Top?

- $\top$
- Interpreted as the full element
- $\Sigma_\top^*$ in **LANG**
  This is the full language
- $X \times X$ in **REL**
  This relates every element in X to every element in X

Problems with Kleene Algebra with Top

- If $\top$ is the full relation in **REL**, then **REL** $\models a + a\top a = a\top a$



- If $\top$ is the full language in **LANG**, then **LANG** $\not\models a + a\top a = a\top a$

$$[\![a + a\top a]\!]_L = \{a, aa, aaa, aba, \dots\}$$
$$[\![a\top a]\!]_L = \{aa, aaa, aba, \dots\}$$

- **KA** with top cannot be sound and complete for all models
- Can **KA** with top be sound and complete for *some* models?

Axiom for KA with Top (1)

- $\top$ is the full element
- $e + \top = \top$
  $[\![e + \top]\!]_{\mathsf{L}} = [\![e]\!]_{\mathsf{L}} \cup [\![\top]\!]_{\mathsf{L}} = [\![\top]\!]_{\mathsf{L}} = \Sigma_\top^*$
- $e \leq f$ is short for $e + f = f$
- If $e \leq f$, then $[\![e]\!]_{\mathsf{L}} \subseteq [\![f]\!]_{\mathsf{L}}$
- $e \leq \top$ holds for all e
- What happens if $e \leq \top$ is added to **KA**?

Axiom for KA with Top (2)

- Axiom T: $e \leq \top$
- $\mathbf{KA}_T$ is Kleene algebra with axiom T
- $\mathbf{KA}_T$ is sound and complete for **LANG**
- Soundness is fairly trivial

Axiom for KA with Top (3)

Definition (Language closure C_T)

Given $u, v \in \Sigma_\top^*$, we write $u \leftsquigarrow_T v$, if u is obtained by replacing some $\top$ in v with an arbitrary word $w \in \Sigma_\top^*$.

$$C_T(L) = \{u \mid u \leftsquigarrow_T^* v \textit{ for some } v \in L\}$$

Definition (Expression closure r)

Let r be the unique homomorphism on regular expressions with $\top$, such that:

$r(a) = a$, for all letters $a \in \Sigma$

$r(\top) = (a + b + \cdots + \top)^*$

Kleene Algebra with Top for LANG

$\mathbf{LANG} \vDash e = f$	
$\Rightarrow C_T[\![e]\!]_\mathsf{L} = C_T[\![f]\!]_\mathsf{L}$	$C_T[\![\cdot]\!]_\mathsf{L}$ is a member of **LANG**
$\Leftrightarrow [\![r(e)]\!]_\mathsf{L} = [\![r(f)]\!]_\mathsf{L}$	$[\![r(e)]\!]_\mathsf{L} = C_T[\![e]\!]_\mathsf{L}$
$\Leftrightarrow \mathbf{KA} \vdash r(e) = r(f)$	completeness of **KA**
$\Rightarrow \mathbf{KA}_T \vdash e = f$	$\mathbf{KA}_T \vdash e = r(e)$
$\Rightarrow \mathbf{LANG} \vDash e = f$	soundness of $\mathbf{KA}_T$

Kleene Algebra with Top for REL

- **REL** $\models e = f \not\Leftrightarrow$ **LANG** $\models e = f$
- **LANG** $\not\models a + a\top a = a\top a$ but **REL** $\models a + a\top a = a\top a$
- **KA**$_T$ is sound w.r.t. **LANG**
- **KA**$_T$ is not complete w.r.t. **REL**
- What axiom needs to be added to **KA**$_T$ to make it (more) complete w.r.t. **REL**?
- Axiom F: $e \leq e \cdot \top \cdot e$
- **KA**$_F$ is Kleene algebra with axioms T and F
- **KA**$_F$ is sound and complete w.r.t. **REL**

Kleene Algebra with Top for REL

$\mathbf{REL} \vDash e = f$	
$\Rightarrow C_F[\![e]\!]_{\mathsf{L}} = C_F[\![f]\!]_{\mathsf{L}}$	C_T, with $w \top w \mapsto w$
$\Leftrightarrow E(C_T[\![e]\!]_{\mathsf{L}}) = E(C_T[\![f]\!]_{\mathsf{L}})$	$C_F = E \circ C_T$
$\Leftrightarrow [\![s(r(e))]\!]_{\mathsf{L}} = [\![s(r(f))]\!]_{\mathsf{L}}$	s on expressions $=$ E on languages
$\Leftrightarrow \mathbf{KA} \vdash s(r(e)) = s(r(f))$	completeness of **KA**
$\Rightarrow \mathbf{KA}_F \vdash e = f$	$\mathbf{KA}_T \vdash e = r(e)$ and $\mathbf{KA}_F \vdash e = s(e)$
$\Rightarrow \mathbf{REL} \vDash e = f$	soundness of $\mathbf{KA}_F$ w.r.t. **REL**

Summary of part II

- $\top$ is the full element
- Axiom T: $e \leq \top$ i.e. $e + \top = \top$
- Axiom F: $e \leq e \cdot \top \cdot e$ i.e. $e + e \cdot \top \cdot e = e \cdot \top \cdot e$
  This is not sound for **LANG**
- $\mathbf{KA}_T$ is sound and complete w.r.t. **LANG**
- $\mathbf{KA}_F$ is sound and complete w.r.t. **REL**

Thank you!

Tobias Kappé.
Master of logic course notes for kleene algebra, 2022.

Damien Pous and Jana Wagemaker.
Completeness theorems for kleene algebra with top.
In Bartek Klin, Slawomir Lasota, and Anca Muscholl, editors, *33rd International Conference on Concurrency Theory, CONCUR 2022, September 12-16, 2022, Warsaw, Poland*, volume 243 of *LIPIcs*, pages 26:1–26:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.