

A Dependency Pair Framework for Innermost Complexity Analysis of Term Rewrite Systems

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September 12, 2025

Term Rewriting Systems

definition

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Rules: generate a relation between terms.

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Rules: generate a relation between terms.

Example:

Function symbols: **Plus** : 2, **Times** : 2, **S** : 1, **0** : 0.

Rules:

$$\begin{aligned}\text{Plus}(x, 0) &\rightarrow x \\ \text{Plus}(x, \text{S}(y)) &\rightarrow \text{S}(\text{Plus}(x, y)) \\ \text{Times}(x, 0) &\rightarrow 0 \\ \text{Times}(x, \text{S}(y)) &\rightarrow \text{Plus}(x, \text{Times}(x, y))\end{aligned}$$

Term Rewriting Systems

example

$$\begin{aligned}\text{Plus}(x, 0) &\rightarrow x \\ \text{Plus}(x, \text{S}(y)) &\rightarrow \text{S}(\text{Plus}(x, y))\end{aligned}$$

Example reduction:

Plus(0, S(0))

Term Rewriting Systems

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$$\begin{aligned}\text{Plus}(x, 0) &\rightarrow x \\ \text{Plus}(x, \text{S}(y)) &\rightarrow \text{S}(\text{Plus}(x, y)) \quad \Leftarrow\end{aligned}$$

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$$\underline{\text{Plus}}(0, \text{S}(0)) \rightarrow \text{S}(\underline{\text{Plus}}(0, 0))$$

Term Rewriting Systems

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$$\begin{aligned}\text{Plus}(x, 0) &\rightarrow x && \Leftarrow \\ \text{Plus}(x, \text{S}(y)) &\rightarrow \text{S}(\text{Plus}(x, y))\end{aligned}$$

Example reduction:

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Example reduction:

$$\underline{\text{Plus}}(0, \text{S}(0)) \rightarrow \text{S}(\underline{\text{Plus}}(0, 0)) \rightarrow \text{S}(0)$$

Runtime complexity

Main Question

How long does a computation take (using innermost reduction)?

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Example: `Plus(s, t):`

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Example: `Times`(*s*, *t*):

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Example: `Times`(*s*, *t*): $\mathcal{O}(n^2)$

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Idea: consider **function calls**

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Investigating termination and complexity

Idea: consider **function calls**

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List all the calls for each rule:

$$\begin{aligned}\text{Plus}^\sharp(x, 0) &\rightarrow \langle \rangle \\ \text{Plus}^\sharp(x, \text{S}(y)) &\rightarrow \langle \text{Plus}^\sharp(x, y) \rangle \\ \text{Times}^\sharp(x, 0) &\rightarrow \langle \rangle \\ \text{Times}^\sharp(x, \text{S}(y)) &\rightarrow \langle \text{Plus}^\sharp(x, \text{Times}^\sharp(x, y)) ; \\ &\quad \text{Times}^\sharp(x, y) \rangle\end{aligned}$$

Various methods!

- usable rules
- reduction pairs
- leaf removal
- knowledge propagation
- narrowing