Lambda-Calculus and Type Theory
ISR 2024
Obergurgl, Austria
Herman Geuvers & Niels van der Weide
Radboud University Nijmegen NL
Exercises Day 1

Lecture 1. Introduction, syntax and operational semantics of untyped lambda calculus

1. Make exercises 2.5 – 2.10 of *Introduction to Lambda Calculus - selected pages* by Barendregt and Barendsen.

Lecture 2. Simple type theory, formulas-as-types and proofs-as-terms

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides of the lecture.
Do the exercises that are approproate for your level.

1. Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\,z) : (\beta\to\alpha)\to((\beta\to\alpha)\to\alpha)\to\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\,z)$ with type information in such a way that it is of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$

   (c) Give a "simpler" term of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$.

3. (a) Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma\to\varepsilon.\lambda y{:}(\gamma\to\varepsilon)\to\varepsilon.y(\lambda z{:}\gamma.y\,x) : (\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   (b) Give another term of the same type

$$(\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

   and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

4. In all of the following cases: give a typing derivation.

   (a) Find a term of type $(\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$

(b) Find two terms of type $(\delta\to\delta\to\alpha)\to(\gamma\to\alpha)\to(\alpha\to\beta)\to\delta\to\gamma\to\beta$

(c) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\alpha$

5. Consider the following term "with holes" N, where $A = \alpha\to\alpha$ and $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ are copies of the well-known λ-term $\mathbf{I}$ ($:= \lambda x.x$).

$$N \qquad := \qquad \lambda y{:}?.(\lambda x{:}A\to A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

Fill in the type for ? in N, give the types for $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ and give the type of N itself in simple type theory ($\lambda\to$) à la Church. (Note that A abbreviates $\alpha\to\alpha$.)

Lecture 3. First order dependent type theory, formulas-as-types and proofs-as-terms

NB. $\to$ binds strongest.

1. Give a precise derivation of the following judgment.

$$A : *,\ P : A \to *,\ a : A \vdash (P\,a) \to * : \square$$

(Advise: give the derivation in "flag style", as it was shown in the lecture.)

2. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x\to Q\,x)\to(\Pi x{:}A.P\,x)\to\Pi x{:}A.Q\,x$$

Do this by giving a derivation in "flag style", where you may omit derivations of the well-formedness of types.

3. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x\to\Pi z{:}A.R\,x\,z)\to(\Pi x{:}A.P\,x)\to\Pi z{:}A.R\,z\,z.$$

(**NB**. Read this type in the proper way: $\to$ binds stronger than Π!)

4. Give a term M of type $\Pi x{:}A.P(f(f\,x))$ in the context

$$\begin{aligned}\Gamma \quad := \quad & A : *, P : A\to *, f : A\to A, g : A\to A,\\ & h : \Pi x{:}A.P(f\,x)\to P(g\,x), k : \Pi x, y{:}A.(P\,x\to P\,y)\to P(f\,x).\end{aligned}$$

Also give a derivation of $\Gamma \vdash M : \Pi x{:}A.P(f(f\,x))$ in 'short form', so you don't have to show the well-formedness of the types.

5. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x\to Q)\to(\Pi x{:}A.P\,x)\to Q$$

What is special about your context? Explain how your context explicitly ensures a property for the type A.

6. Find a term from the given hypotheses of the following type and write down the context in which this term is typed.

$$\begin{aligned}
&\forall x.\,(P(x) \to R(x, f(x))),\\
&\forall x, y.\,(R(x, y) \to R(y, x)),\\
&\forall x, y.\,(R(x, y) \to R(f(y), x)) \quad \vdash \quad \forall x.\,(P(x) \to R(f(x), f(x)))
\end{aligned}$$