Lambda-Calculus and Type Theory
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Obergurgl, Austria
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Exercises Day 1

Lecture 1. Introduction, syntax and operational semantics of untyped lambda calculus

1. Make exercises 2.5 – 2.10 of *Introduction to Lambda Calculus - selected pages* by Barendregt and Barendsen.

Lecture 2. Simple type theory, formulas-as-types and proofs-as-terms

See the course notes – notably *Introduction to Type Theory* by Herman Geuvers – and the slides of the lecture.
Do the exercises that are approproate for your level.

1. Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\alpha\to\beta}.\lambda y^{\beta\to\gamma}.\lambda z^{\alpha}.y(xz) : (\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$$

2. (a) Verify in detail (by giving a derivation in $\lambda\!\to$) that

$$\lambda x^{\beta\to\alpha}.\lambda y^{(\beta\to\alpha)\to\alpha}.y(\lambda z^{\beta}.x\,z) : (\beta\to\alpha)\to((\beta\to\alpha)\to\alpha)\to\alpha$$

   (b) "Dress up" the λ-term $\lambda x.\lambda y.y(\lambda z.x\,z)$ with type information in such a way that it is of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$
   Answer: ....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y(\lambda z{:}\beta.x\,z)$
   End Answer ....

   (c) Give a "simpler" term of type $(\beta\to\gamma)\to((\beta\to\gamma)\to\alpha)\to\alpha$.
   Answer: ....
   Here is the term without typing derivation.
   $\lambda x{:}\beta\to\gamma.\lambda y{:}(\beta\to\gamma)\to\alpha.y\,x$
   End Answer ....

3. (a) Give the natural deduction (either in Fitch style or in tree form) that corresponds to

$$\lambda x{:}\gamma\to\varepsilon.\lambda y{:}(\gamma\to\varepsilon)\to\varepsilon.y(\lambda z{:}\gamma.y\,x) : (\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

(b) Give another term of the same type

$$(\gamma\to\varepsilon)\to((\gamma\to\varepsilon)\to\varepsilon)\to\varepsilon$$

and the natural deduction (either in Fitch style or in tree form) that it corresponds to.

4. In all of the following cases: give a typing derivation.

(a) Find a term of type $(\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$
Answer: ....
Finding a term is best done by finding a derivation of (a term of) this type as a formula. Call $\sigma := (\delta\to\delta\to\alpha)\to(\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$

1	$x : \delta\to\delta\to\alpha$	
2	$y : \alpha\to\beta\to\gamma$	
3	$z : \delta\to\beta$	
4	$v : \delta$	
5	$x\,v : \delta\to\alpha$	app, 1, 4
6	$x\,v\,v : \alpha$	app, 5, 4
7	$y(x\,v\,v) : \beta\to\gamma$	app, 2, 6
8	$(z\,v) : \beta$	app, 3, 4
9	$y(x\,v\,v)(z\,v) : \gamma$	app, 2, 6
10	$\lambda v : \delta.y(x\,v\,v)(z\,v) : \delta\to\gamma$	λ-rule, 4, 9
11	$\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, 10
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, 11
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.y(x\,v\,v)(z\,v) : \sigma$	λ-rule, 1, 12

This term is created by filling in the ? in the following “template”

1	$x : \delta\to\delta\to\alpha$	
2	$y : \alpha\to\beta\to\gamma$	
3	$z : \delta\to\beta$	
4	$v : \delta$	
5	$\ldots$	
6	$\ldots$	
7	$\ldots$	
8	$\ldots$	
9	$? : \gamma$	
10	$\lambda v : \delta.? : \delta\to\gamma$	λ-rule, 4, .
11	$\lambda z : \delta\to\beta.\lambda v : \delta.? : (\delta\to\beta)\to\delta\to\gamma$	λ-rule, 3, .
12	$\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : (\alpha\to\beta\to\gamma)\to(\delta\to\beta)\to\delta\to\gamma$	λ-rule, 2, .
13	$\lambda x : \delta\to\delta\to\alpha.\lambda y : \alpha\to\beta\to\gamma.\lambda z : \delta\to\beta.\lambda v : \delta.? : \sigma$	λ-rule, 1, .

The $? : \gamma$ should clearly be of the form $y ?_1 ?_2$ with $?_1 : \alpha$ and $?_2 : \beta$ ... and so forth. So one basically works "inside out" constructing the term. (This is basically "goal directed theorem proving".)
End Answer ....

(b) Find two terms of type $(\delta\to\delta\to\alpha)\to(\gamma\to\alpha)\to(\alpha\to\beta)\to\delta\to\gamma\to\beta$
Answer: ....
Here are the terms, construct the derivations yourself.
$\lambda x : \delta\to\delta\to\alpha.\lambda y : \gamma\to\alpha.\lambda z : \alpha\to\beta.\lambda v : \delta.\lambda w : \gamma.z\,(x\,v\,v)$
$\lambda x : \delta\to\delta\to\alpha.\lambda y : \gamma\to\alpha.\lambda z : \alpha\to\beta.\lambda v : \delta.\lambda w : \gamma.z\,(y\,w)$
End Answer ....

(c) Find a term of type $((\alpha\to\beta)\to\alpha)\to(\alpha\to\alpha\to\beta)\to\alpha$
Answer: ....
Here is the term, construct the derivation yourself. $\lambda f : (\alpha\to\beta)\to\alpha.\lambda g : \alpha\to\alpha\to\beta.f(\lambda x{:}\alpha.g\,x\,x)$
End Answer ....

5. Consider the following term "with holes" N, where $A = \alpha\to\alpha$ and $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ are copies of the well-known λ-term $\mathbf{I}$ ($:= \lambda x.x$).

$$N \qquad := \qquad \lambda y{:}?.(\lambda x{:}A\to A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

Fill in the type for ? in N, give the types for $\mathbf{I}_1$ and $\mathbf{I}_2$ and $\mathbf{I}_3$ and $\mathbf{I}_4$ and give the type of N itself in simple type theory ($\lambda\to$) à la Church. (Note that A abbreviates $\alpha\to\alpha$.)
Answer: ....
$\mathbf{I}_2 : A\to A$ and $\mathbf{I}_4 : A$ (because $\mathbf{I}_4$ is the argument of x), so $\mathbf{I}_3\,y : \alpha$ (because it is the argument of $x\,\mathbf{I}_4 : A$) and hence $\mathbf{I}_3 : A$ and $y : \alpha$, so $? = \alpha$. We

have $x\,\mathbf{I}_4\,(\mathbf{I}_3\,y) : \alpha$, so $\mathbf{I}_1 : A$ and $\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)) : \alpha$. The type of N is $\alpha{\to}\alpha$, so A.
End Answer . . . .

Lecture 3. First order dependent type theory, formulas-as-types and proofs-as-terms

NB. $\to$ binds strongest.

1. Give a precise derivation of the following judgment.

$$A : *,\ P : A \to *,\ a : A \vdash\ (P\,a) \to * : \Box$$

(Advise: give the derivation in "flag style", as it was shown in the lecture.)

Answer: . . . .
We give it completely, using the $\to$-formation rule as a degenerate case of the Π-formation rule (if $x \notin \mathrm{FV}(B)$):

$$\frac{\Gamma \vdash A : * \quad \Gamma \vdash B : *}{\Gamma \vdash A \to B : *}\ \to\text{-form} \qquad \frac{\Gamma \vdash A : * \quad \Gamma, x{:}A \vdash B : *}{\Gamma \vdash \Pi x{:}A.B : *}\ \Pi\text{-form}$$

$$\begin{array}{r|l|l|l|l l}
1 & * : \Box & & & & \\
2 & & A : * & & & \text{var, 1} \\
3 & & A \to * : \Box & & & \to\text{-form, 2, 1} \\
4 & & & P : A \to * & & \text{var, 3} \\
5 & & & & a : A & \text{var, 2} \\
6 & & & & P\,a : * & \text{app, 4, 5} \\
7 & & & & P\,a \to * : \Box & \to\text{-form, 6, 1}
\end{array}$$

End Answer . . . .

2. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x{\to}Q\,x){\to}(\Pi x{:}A.P\,x){\to}\Pi x{:}A.Q\,x$$

Do this by giving a derivation in "flag style", where you may omit derivations of the well-formedness of types.

Answer: . . . .
Write σ for $(\Pi x{:}A.P\,x{\to}Q\,x){\to}(\Pi x{:}A.P\,x){\to}\Pi x{:}A.Q\,x$.

1	$A : *$	
2	$P : A \to *$	
3	$Q : A \to *$	
4	$h : \Pi x{:}A.P\,x{\to}Q\,x$	
5	$g : \Pi x{:}A.P\,x$	
6	$x : A$	
7	$h\,x : P\,x \to Q\,x$	app, 4, 6
8	$g\,x : P\,x$	app, 5, 6
9	$h\,x(g\,x) : Q\,x$	app, 7, 8
10	$\lambda x{:}A.h\,x(g\,x) : \Pi x{:}A.Q\,x$	λ-rule, 6, 9
11	$\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : (\Pi x{:}A.P\,x) \to \Pi x{:}A.Q\,x$	λ-rule, 5, 10
12	$\lambda h{:}\Pi x{:}A.P\,x{\to}Q\,x.\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : \sigma$	λ-rule, 4, 11

So:

$$A : *, P : A \to *, Q : A \to * \vdash \lambda h{:}\Pi x{:}A.P\,x{\to}Q\,x.\lambda g{:}\Pi x{:}A.P\,x.\lambda x{:}A.h\,x(g\,x) : \sigma$$

End Answer ....

3. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z){\to}(\Pi x{:}A.P\,x){\to}\Pi z{:}A.R\,z\,z.$$

(**NB**. Read this type in the proper way: $\to$ binds stronger than Π!)

Answer: ....
We write τ for $(\Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z){\to}(\Pi x{:}A.P\,x){\to}\Pi z{:}A.R\,z\,z)$. We only give the proof term, not the derivation.

$A : *, P : A \to *, R : A \to A \to * \vdash$
$\lambda h : \Pi x{:}A.P\,x{\to}\Pi z{:}A.R\,x\,z.\lambda g : \Pi x{:}A.P\,x.\lambda y : A.h\,y(g\,y)\,y : \tau$

End Answer ....

4. Give a term M of type $\Pi x{:}A.P(f(f\,x))$ in the context

$$\begin{aligned}\Gamma \quad := \quad & A : *, P : A{\to}*, f : A{\to}A, g : A{\to}A,\\ & h : \Pi x{:}A.P(f\,x){\to}P(g\,x), k : \Pi x, y{:}A.(P\,x{\to}P\,y){\to}P(f\,x).\end{aligned}$$

Also give a derivation of $\Gamma \vdash M : \Pi x{:}A.P(f(f\,x))$ in 'short form', so you don't have to show the well-formedness of the types.

Answer: ....
Only the term:

$$\lambda x : A.k(f\,x)(g\,x)(h\,x)$$

End Answer ....

5. Find a term of the following type and write down the context in which this term is typed.

$$(\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$$

What is special about your context? Explain how your context explicitly ensures a property for the type A.

Answer: ....

$A : *, P : A \to *, Q : *, \underline{a : A} \vdash$
$\lambda h{:}\Pi x{:}A.P\,x{\to}Q.\lambda g{:}\Pi x{:}A.P\,x.h\,a\,(g\,a) : (\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$

We need a declaration of a variable $a : A$ in the context, stating that A is not empty. If we don't have a term of type A, we can not construct a term of this type, so if A is just a variable in the context, teh only thing we can do is to decalre $a : A$ as well.

Note that, if A is the "empty type", the type $(\Pi x{:}A.P\,x{\to}Q){\to}(\Pi x{:}A.P\,x){\to}Q$, interpreted as a formula states something that is just not true:
$\forall x{:}A.P\,x{\to}Q$ and $\forall x{:}A.P\,x$ are both vacuouslu try of A is empty, but Q need not be.
End Answer ....

6. Find a term from the given hypotheses of the following type and write down the context in which this term is typed.

$$\begin{array}{l}\forall x.\,(P(x) \to R(x, f(x))),\\ \forall x, y.\,(R(x, y) \to R(y, x)),\\ \forall x, y.\,(R(x, y) \to R(f(y), x)) \quad \vdash \quad \forall x.\,(P(x) \to R(f(x), f(x)))\end{array}$$

Answer: ....
We only give the term, type and context.

In context $D : *, f : D \to D, P : D \to *, R : D \to D \to *,$
$t : \Pi x{:}D.P\,x \to R\,x\,(f\,x),$
$s : \Pi x, y{:}D.R\,x\,y \to R\,y\,x,$
$q : \Pi x, y{:}D.R\,x\,y \to R\,(f\,y)\,x$, we have

$$\lambda x{:}D.\lambda h{:}P\,x.\,q\,(f\,x)\,x\,(s\,x\,(f\,x)\,(t\,x\,h)) : \Pi x{:}D.P\,x \to R\,(f\,x)\,(f\,x).$$

End Answer . . . .