

lecture 9. Church-Rosser property

All exercises are about the Church-Rosser proof that is on the slides (due to Takahashi) and that we have presented at the lecture. We recall the definition of \Rightarrow using derivation rules:

$$\frac{}{x \Rightarrow x} \text{ (var)} \qquad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'} \text{ (\lambda)}$$

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{MN \Rightarrow M'N'} \text{ (app)} \qquad \frac{M \Rightarrow M' \quad N \Rightarrow N'}{(\lambda x.M)N \Rightarrow M'[N'/x]} \text{ (\beta)}$$

1. Consider the term $M = (\lambda x y.x x(x y))(\mathbf{II})$
 - (a) Give the reduction graph of M . (You may abbreviate \mathbf{II} to J and $\lambda x y.x x(x y)$ to P .)
 - (b) Compute M^* and $(M^*)^*$.
 - (c) Prove that $M \Rightarrow M^*$ and $M^* \Rightarrow (M^*)^*$ by giving a derivation.
2. In the definition of \Rightarrow , we change clause (β) into

$$\frac{M \Rightarrow \lambda x.P \quad N \Rightarrow N'}{MN \Rightarrow P[N'/x]}$$

- (a) Give the definition of $(-)^*$ that goes with this adapted definition of \Rightarrow .
 - (b) Prove again (with these adapted definitions) that $M \Rightarrow N$ implies $N \Rightarrow M^*$, by doing the inductive step for case (β) .
3. The η -reduction rule is: $\lambda x.M x \rightarrow_\eta M$, if $x \notin \text{FV}(M)$. In order to prove CR for $\beta\eta$ we add a clause for η -redexes to the definition of \Rightarrow :

$$\frac{M \Rightarrow M'}{\lambda x.M x \Rightarrow M'} \quad x \notin \text{FV}(M)$$

- (a) Show that now $(\lambda y x.y x)\mathbf{I} \Rightarrow \mathbf{I}$, and show that in the original definition, this is not the case.
- (b) Define $(-)^*$ for this extension to η