Lambda-Calculus and Type Theory
ISR 2024
Obergurgl, Austria
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Exercises Day 4

lecture 9. Church-Rosser property

All exercises are about the Church-Rosser proof that is on the slides (due to Takahashi) and that we have presented at the lecture. We recall the definition of $\Rightarrow$ using derivation rules:

$$\frac{}{x \Rightarrow x}\,(\mathrm{var}) \qquad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}\,(\lambda)$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N'}{M\,N \Rightarrow M'\,N'}\,(\mathrm{app}) \qquad \frac{M \Rightarrow M' \qquad N \Rightarrow N'}{(\lambda x.M)\,N \Rightarrow M'[N'/x]}\,(\beta)$$

1. Consider the term $M = (\lambda x\,y.x\,x(x\,y))(\mathbf{I}\,\mathbf{I})$
   (a) Give the reduction graph of M. (You may abbreviate $\mathbf{I}\,\mathbf{I}$ to J and $\lambda x\,y.x\,x(x\,y)$ to P.)
   (b) Compute M^* and $(M^*)^*$.
   (c) Prove that $M \Rightarrow M^*$ and $M^* \Rightarrow (M^*)^*$ by giving a derivation.
2. In the definition of $\Rightarrow$, we change clause (β) into

$$\frac{M \Rightarrow \lambda x.P \qquad N \Rightarrow N'}{M\,N \Rightarrow P[N'/x]}$$

   (a) Give the definition of $(-)^*$ that goes with this adapted definition of $\Rightarrow$.
   (b) Prove again (with these adapted definitions) that $M \Rightarrow N$ implies $N \Rightarrow M^*$, by doing the inductive step for case (β).
3. The η-reduction rule is: $\lambda x.M\,x \rightarrow_\eta M$, if $x \notin \mathrm{FV}(M)$. In order to prove CR for $\beta\eta$ we add a clause for η-redexes to the definition of $\Rightarrow$:

$$\frac{M \Rightarrow M'}{\lambda x.M\,x \Rightarrow M'}\; x \notin \mathrm{FV}(M)$$

   (a) Show that now $(\lambda yx.y\,x)\mathbf{I} \Rightarrow \mathbf{I}$, and show that in the original definition, this is not the case.
   (b) Define $(-)^*$ for this extension to η

Answer: ....
For substitution (substitute N for x in M) one sometimes writes $M[x := N]$ (e.g. Takahashi) and sometimes $M[N/x]$ (the exercise sheet).

Exercise Church-Rosser

1

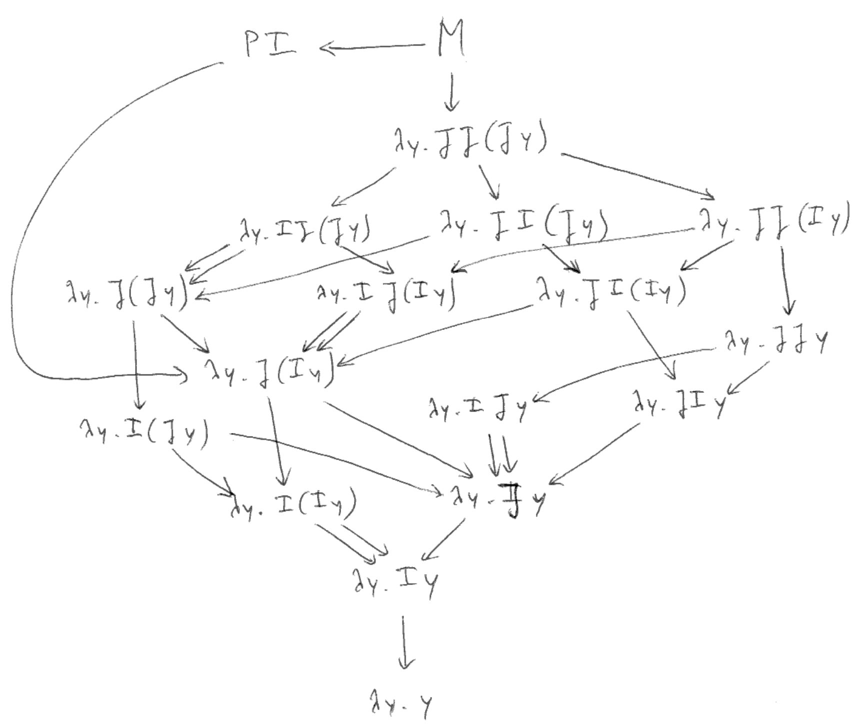
(1) $M = (\lambda xy.\, xx(xy))(II)$

$P = \lambda xy.\, xx(xy)$

$J = II$

(a) We omit duplicates in the graph.

Note that $\lambda y.\, II(xI)$ is just $\lambda y.\, J(xI)$ etc.



(b) $M^* = \lambda y.\, II(Iy)$

$(M^*)^* = \lambda y.\, Iy$

1c

2

$$\boxed{M \Rightarrow M^*}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x \Rightarrow x \quad x \Rightarrow x}{xx \Rightarrow xx} \qquad \dfrac{x \Rightarrow x \quad y \Rightarrow y}{xy \Rightarrow xy}}{xx(xy) \Rightarrow xx(xy)}}{\lambda y.xx(xy) \Rightarrow \lambda y.xx(xy)} \qquad \dfrac{z \Rightarrow z \quad \lambda z.z \Rightarrow \lambda z.z}{(\lambda z.z)I \Rightarrow I}}{(\lambda xy.xx(xy))(II) \Rightarrow \lambda y.II(Iy)}$$

$$\boxed{I = \lambda z.z} \qquad \dfrac{\dfrac{\dfrac{z \Rightarrow z \quad \dfrac{z \Rightarrow z}{I \Rightarrow I}}{II \Rightarrow I} \qquad \dfrac{z \Rightarrow z \quad y \Rightarrow y}{Iy \Rightarrow y}}{II(Iy) \Rightarrow Iy}}{\lambda y.II(Iy) \Rightarrow \lambda y.Iy} \qquad \boxed{I = \lambda z.z}$$

$$\boxed{M^* \Rightarrow (M^*)^*}$$

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2 (a) Change the cases (β_3^*) and (β_4^*) to the following

(β_5^*) If $M_1^* \equiv \lambda x.P$, then $(M_1 M_2)^* \equiv P[x := M_2^*]$

(b)
$$\dfrac{M \Rightarrow \lambda x.P \qquad N \Rightarrow N'}{MN \Rightarrow \cancel{P[N'/x]}\ P[x := N']}$$

IH $\lambda x.P \Rightarrow M^*$, $N' \Rightarrow N^*$

To prove: $\cancel{P[N'/x] \Rightarrow (MN)^*}$ $P[x := N'] \Rightarrow (MN)^*$

From $\lambda x.P \Rightarrow M^*$ we deduce that $M^* = \lambda x.Q$ with $P \Rightarrow Q$

So $(MN)^* = Q[x := N^*]$.

we have $P \Rightarrow Q$ and $N' \Rightarrow N^*$, so by substitution (property (3) in the Takahashi paper) we conclude $P[x := N'] \Rightarrow Q[x := N^*]$

$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad (MN)^* \quad \boxtimes$$

(3) (a) (3)

$$\dfrac{\dfrac{y \Rightarrow y}{\lambda x.\, y\, x \Rightarrow y} \qquad I \Rightarrow I}{(\lambda \cancel{y}\, \lambda x.\, y\, x)\, I \Rightarrow I}$$

$$I \equiv y\,[y := I]$$

We don't have $(\lambda y.\, \lambda x.\, y\, x)\, I \Rightarrow I$

because if this were derivable, a derivation has to have the following shape

$$\dfrac{\dfrac{\circledast}{\lambda x.\, y\, x \Rightarrow M'} \qquad I \Rightarrow I}{(\lambda y.\, \lambda x\, y\, x)\, I \Rightarrow M'[y := I]}$$

$$M'[y := I] \equiv I$$

Note $M'[y = I] \equiv I$ can be because of

(i) $M' \equiv y$

(ii) $M' \equiv I$

In case (i) we must have a derivation $\dfrac{\circledast}{\lambda x.\, y\, x \Rightarrow y}$

but that can't be, because a λ-abstraction only parallel reduces to another λ-abstraction

In case (ii) we must have $\dfrac{\circledast}{\lambda x\, y x \Rightarrow \lambda x.\, x}$

and so $y\, x \Rightarrow x$, which is not the case either.

So: we can't have such a derivation ⊠

(b) $(\lambda x.\, M)^* = \begin{cases} P^* & \text{if } M = P x \text{ with } x \notin FV(P) \\ \lambda x.\, M^* & \text{otherwise.} \end{cases}$

End Answer . . . .