Lambda-Calculus and Type Theory
ISR 2024
Obergurgl, Austria
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Exercises Day 5

Lecture 10. Weak normalization and strong normalization

1. In the proof of WN for $\lambda\!\to$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1.$

   Prove that this is the same as taking as the second clause

   - $h(\sigma \to \tau) := \max(h(\sigma) + 1, h(\tau)).$

2. Consider the following term $N : A$, where $A = \alpha \to \alpha$ and $\mathbf{I}_1 : A$ and $\mathbf{I}_2 : A \to A$ and $\mathbf{I}_3 : A$ and $\mathbf{I}_4 : A$ are copies of the well-known λ-term $\mathbf{I}$ ($:= \lambda x.x$).

   $$N \quad := \quad \lambda y{:}\alpha.(\lambda x{:}A \to A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

   (a) Determine $m(N)$, the *measure* of N as defined in the weak normalization proof.

   (b) Determine which redex will be contracted following the strategy in the weak normalization proof.

3. In the proof of WN for $\lambda\!\to$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that does not contain any other redex of height $h(M)$, then this does not create a new redex of maximum height.

   Show that this holds for the case

   $$\begin{array}{rcll} M & = & (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})) & \\ & \longrightarrow_\beta & (\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))(\lambda v : B.(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))\,\mathbf{I}) & = \quad P \end{array}$$

   where $B = \alpha \to \alpha$, $C = B \to B$ and $A = C \to B$.

   Also show that $m(M) >_l m(P)$.

4. Suppose X, Y, and Z are properties of λ-terms. Then we can have the following situations: If M satisfies property X and N satisfies property Y, then

   (a) Yes, property Z always holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \in Z)$

   (b) No, property Z never holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \notin Z)$

(c) Undec, property Z holds for some M, N, and doesn't hold for some other M, N (so $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \in Z$ and $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \notin Z)$

Fill in the following diagram with Yes, No and Undec and motivate your answers. In case of Undec, give M, N for both cases.

	$N \in \mathsf{WN}$	$N \in \neg\mathsf{SN}$
$M \in \mathsf{WN}$	$M\,N \in \mathsf{WN}$?	$M\,N \in \mathsf{SN}$?
$M \in \mathsf{SN}$	$M\,N \in \mathsf{SN}$?	$M\,N \in \neg\mathsf{SN}$?

5. Prove that *type reduction* is SN for $\lambda 2$ a la Church. (Define a simple measure on terms that decreases with type reduction.)

Lecture 11. Normalization by Evaluation

1. In the lecture, we discussed normalization by evaluation for monoid expressions. Use the algorithm on slides 17 and 18 to normalize the expression

$$(v(5) \cdot u) \cdot (u \cdot v(3)) \cdot (u \cdot v(2) \cdot u).$$

2. The normalization algorithm on slides 17 and 18 returns normal forms of the shape $v(x_1) \cdot \ldots \cdot v(x_n) \cdot u$. Modify this algorithm so that it returns normal forms of the shape $v(x_1) \cdot \ldots \cdot v(x_n)$ instead.

3. Construct the functions

$$u_A^\Gamma : \mathsf{NE}_A(\Gamma) \to [\![A]\!](\Gamma)$$

$$q_A^\Gamma : [\![A]\!](\Gamma) \to \mathsf{NF}_A(\Gamma)$$

Hint: define them by mutual recursion and use induction on A.