

**Lecture 10. Weak normalization and strong normalization**

1. In the proof of WN for  $\lambda \rightarrow$ , the height of a type  $h(\sigma)$  is defined by

- $h(\alpha) := 0$
- $h(\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha) := \max(h(\sigma_1), \dots, h(\sigma_n)) + 1$ .

Prove that this is the same as taking as the second clause

- $h(\sigma \rightarrow \tau) := \max(h(\sigma) + 1, h(\tau))$ .

**Answer:** .....  
 Recall that  $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha$  should be read as  $(\sigma_1 \rightarrow (\dots (\sigma_n \rightarrow \alpha) \dots))$  and this means that every type  $\sigma \rightarrow \tau$  can be written as  $\sigma \rightarrow \tau_1 \dots \tau_n \rightarrow \alpha$  for some  $\alpha$ .

We have two definitions of  $h$  that we call  $h_1$  and  $h_2$  for now:

- $h_1(\alpha) := 0$ ,
- $h_1(\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha) := \max(h_1(\sigma_1), \dots, h_1(\sigma_n)) + 1$ .
- $h_2(\alpha) := 0$ ,
- $h_2(\sigma \rightarrow \tau) := \max(h_2(\sigma) + 1, h_2(\tau))$ .

We prove that  $h_1(\sigma) = h_2(\sigma)$  for all  $\sigma$  by induction on  $\sigma$ . For type variables this is the case. Case  $\sigma = \sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha$ :

$$\begin{aligned}
 h_2(\sigma) &= \max(h_2(\sigma_1) + 1, h_2(\sigma_2 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha)) \\
 &\stackrel{\text{IH}}{=} \max(h_1(\sigma_1) + 1, h_1(\sigma_2 \rightarrow \dots \rightarrow \sigma_n \rightarrow \alpha)) \\
 &= \max(h_1(\sigma_1) + 1, \max(h_1(\sigma_2), \dots, h_1(\sigma_n)) + 1) \\
 &= \max(h_1(\sigma_1), \dots, h_1(\sigma_n)) + 1 \\
 &= h_1(\sigma)
 \end{aligned}$$

**End Answer** .....

2. Consider the following term  $N : A$ , where  $A = \alpha \rightarrow \alpha$  and  $\mathbf{I}_1 : A$  and  $\mathbf{I}_2 : A \rightarrow A$  and  $\mathbf{I}_3 : A$  and  $\mathbf{I}_4 : A$  are copies of the well-known  $\lambda$ -term  $\mathbf{I}$  ( $:= \lambda x.x$ ).

$$N := \lambda y:\alpha. (\lambda x:A \rightarrow A. \mathbf{I}_1 (x \mathbf{I}_4 (\mathbf{I}_3 y))) \mathbf{I}_2$$

(a) Determine  $m(N)$ , the *measure* of  $N$  as defined in the weak normalization proof.

**Answer:** .....

There are three redexes:

- $R_1 := \mathbf{I}_1 (x \mathbf{I}_4 (\mathbf{I}_3 y))$  of height the type of  $\mathbf{I}_1$ , which is  $h(A) = 1$ .
- $R_2 := (\lambda x:A \rightarrow A. \mathbf{I}_1 (x \mathbf{I}_4 (\mathbf{I}_3 y))) \mathbf{I}_2$  of height the type of  $\lambda x:A \rightarrow A. \mathbf{I}_1 (x \mathbf{I}_4 (\mathbf{I}_3 y))$ , which is  $h((A \rightarrow A) \rightarrow \alpha) = 3$ .
- $R_3 := \mathbf{I}_3 y$  of height the type of  $\mathbf{I}_3$ , which is  $h(A) = 1$ .

**NB.** the sub-term  $\mathbf{I}_4 (\mathbf{I}_3 y)$  is not a redex, as a matter of fact, it isn't a sub-term as the brackets should be read as follows:  $(x \mathbf{I}_4) (\mathbf{I}_3 y) !$   
So  $m(N) = (3, 1)$ .

**End Answer** .....

(b) Determine which redex will be contracted following the strategy in the weak normalization proof, obtaining a term  $N'$ .

**Answer:** .....

We contract redex  $R_2$ , obtaining

$$N' := \lambda y:\alpha. \mathbf{I}_1 (\mathbf{I}_2 \mathbf{I}_4 (\mathbf{I}_3 y))$$

**End Answer** .....

(c) Determine  $m(N')$ , the measure of this reduct of  $N$ .

**Answer:** .....

$N'$  still has redexes  $R_1$  and  $R_3$  that have height 1. We have a new redex  $\mathbf{I}_2 \mathbf{I}_4$  whose height is  $h(A \rightarrow A)$  (the height of the type of  $\mathbf{I}_2$ ), which is 2. So  $m(N') = (2, 1)$

**End Answer** .....

3. In the proof of WN for  $\lambda \rightarrow$ , it is stated that, if  $M \rightarrow_\beta N$  by contracting a redex of maximum height,  $h(M)$ , that does not contain any other redex of height  $h(M)$ , then this does not create a new redex of maximum height.

Show that this holds for the case

$$\begin{aligned} M &= (\lambda x : A. x (\lambda v : B. x \mathbf{I})) (\lambda z : C. z (\mathbf{I})) \\ &\rightarrow_\beta (\lambda z : C. z (\mathbf{I})) (\lambda v : B. (\lambda z : C. z (\mathbf{I})) \mathbf{I}) = P \end{aligned}$$

where  $B = \alpha \rightarrow \alpha$ ,  $C = B \rightarrow B$  and  $A = C \rightarrow B$ .

Also show that  $m(M) >_l m(P)$ .

**Answer:** .....

In  $M$  there are two redexes:

- $R_1 := \mathbf{I} \mathbf{I}$ , whose height is the height of the type of the  $\mathbf{I}$  on the left, which is  $C$ . So the height of  $R_1$  is  $h(C)$ .
- $R_2 := (\lambda x : A. x (\lambda v : B. x \mathbf{I})) (\lambda z : C. z (\mathbf{I}))$ , the whole term, whose height is the height of the type of  $\lambda x : A. x (\lambda v : B. x \mathbf{I})$ , which is  $A \rightarrow B$ . So the height of  $R_2$  is  $h(A \rightarrow B)$ .

$h(B) = 1$ ,  $h(C) = 2$  and  $h(A) = 3$ . Redex  $R_2$  is the single redex of maximum height, which is 4, so  $m(M) = (4, 1)$ . Note that redex  $R_1$  has height  $h(C)$  which is 2.

In  $N$  we have two copies of redex  $R_1$  (of height 2) and two new redexes whose height is the height of the type of  $\lambda z : C.z$  (**II**), which is  $C \rightarrow B$ . We have  $h(C \rightarrow B) = 3$ , so we have  $m(N) = (3, 2)$ . We see that  $m(M) = (4, 1) >_l (3, 2) = m(P)$ .

**End Answer** .....

4. Suppose  $X$ ,  $Y$ , and  $Z$  are properties of  $\lambda$ -terms. Then we can have the following situations: If  $M$  satisfies property  $X$  and  $N$  satisfies property  $Y$ , then
- (a) **Yes**, property  $Z$  always holds (so  $\forall M, N (M \in X \wedge N \in Y \Rightarrow M N \in Z)$ )
  - (b) **No**, property  $Z$  never holds (so  $\forall M, N (M \in X \wedge N \in Y \Rightarrow M N \notin Z)$ )
  - (c) **Undec**, property  $Z$  holds for some  $M, N$ , and doesn't hold for some other  $M, N$  (so  $\exists M, N (M \in X \wedge N \in Y \wedge M N \in Z$  and  $\exists M, N (M \in X \wedge N \in Y \wedge M N \notin Z)$ )

Fill in the following diagram with **Yes**, **No** and **Undec** and motivate your answers. In case of **Undec**, give  $M, N$  for both cases.

	$N \in \text{WN}$	$N \in \neg\text{SN}$
$M \in \text{WN}$	$M N \in \text{WN}?$	$M N \in \text{SN}?$
$M \in \text{SN}$	$M N \in \text{SN}?$	$M N \in \neg\text{SN}?$

**Answer:** .....

We have

	$N \in \text{WN}$	$N \in \neg\text{SN}$		
$M \in \text{WN}$	$M N \in \text{WN}$	<b>Undec</b> <sup>1</sup>	$M N \in \text{SN}$	<b>No</b> <sup>2</sup>
$M \in \text{SN}$	$M N \in \text{SN}$	<b>Undec</b> <sup>3</sup>	$M N \in \neg\text{SN}$	<b>Yes</b> <sup>4</sup>

For **Undec**<sup>1</sup> and **Undec**<sup>3</sup> consider  $M, N = \lambda x. x x$  for a negative example and  $M, N = \lambda x. x$  for a positive example.

For **No**<sup>2</sup> and **Yes**<sup>4</sup>, if  $N \in \neg\text{SN}$ , then  $N$  has an infinite reduction path, so  $M N$  has an infinite reduction path (no matter what  $M$  is).

**End Answer** .....

5. Prove that *type reduction* is SN for  $\lambda 2$  a la Church. (Define a simple measure on terms that decreases with type reduction.)

**Answer:** .....

Define the measure  $f(M)$  as  $f(M) :=$  the number of type-abstractions in  $M$ , so recursively:

$$\begin{aligned}
 f(x) &:= 0 \\
 f(PN) &:= f(P) + f(N) \\
 f(\lambda\alpha.N) &:= 1 + f(N) \\
 f(P\sigma) &:= f(P) \\
 f(\lambda x : \sigma.N) &:= f(N) \\
 f(PN) &:= f(P) + f(N)
 \end{aligned}$$

Then we have that if  $M \rightarrow_{\beta} N$  by a type-reduction (i.e. a redex contraction of the form  $(\lambda\alpha.P)\sigma \rightarrow_{\beta} P[\alpha := \sigma]$ ), then  $f(M) > f(N)$ , which is easily proved by induction on  $M$ .

So: type-reduction is strongly normalizing.

**End Answer** .....

## Lecture 11. Normalization by Evaluation

1. In the lecture, we discussed normalization by evaluation for monoid expressions. Use the algorithm on slides 17 and 18 to normalize the expression

$$(v(5) \cdot u) \cdot (u \cdot v(3)) \cdot (u \cdot v(2) \cdot u).$$

2. The normalization algorithm on slides 17 and 18 returns normal forms of the shape  $v(x_1) \cdot \dots \cdot v(x_n) \cdot u$ . Modify this algorithm so that it returns normal forms of the shape  $v(x_1) \cdot \dots \cdot v(x_n)$  instead.
3. Construct the functions

$$u_A^{\Gamma} : \mathbf{NE}_A(\Gamma) \rightarrow \llbracket A \rrbracket(\Gamma)$$

$$q_A^{\Gamma} : \llbracket A \rrbracket(\Gamma) \rightarrow \mathbf{NF}_A(\Gamma)$$

Hint: define them by mutual recursion and use induction on  $A$ .