Lambda-Calculus and Type Theory

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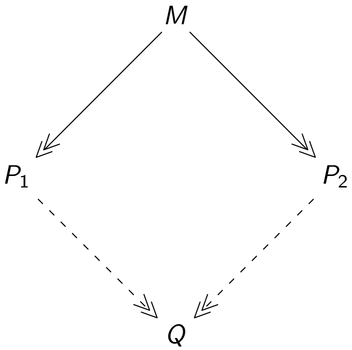
Lecture 9

Church-Rosser property

Todays lecture

- What do we want to prove about type systems?
  Meta Theory
- Church-Rosser (confluence) of reduction

Church-Rosser property, CR



Church-Rosser Theorem for β-reduction, CR_β.
If $M \twoheadrightarrow_\beta P_1$ and $M \twoheadrightarrow_\beta P_2$, then $\exists Q(P_1 \twoheadrightarrow_\beta Q \wedge P_2 \twoheadrightarrow_\beta Q)$

NB. $M \twoheadrightarrow P$ denotes the reflexive transitive closure of $M \rightarrow P$, that is:
$M \twoheadrightarrow P$ iff there is a multi-step (0 or more) reduction from M to P.

We will prove the Church-Rosser Theorem for β-reduction in this lecture.

Church-Rosser (for β) example

$$(\lambda x. y\, x\, x)(\mathbf{I}\, \mathbf{I})$$

General setting: Rewriting systems

DEFINITION A rewriting system is a pair $(A, \rightarrow_R)$, with A a set and $\rightarrow_R \subseteq A \times A$ a relation on A.
Some notation:

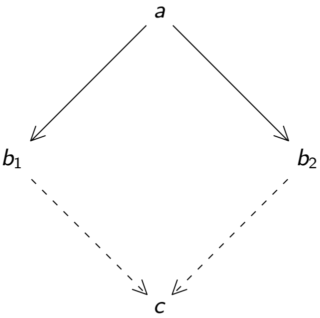
- $a \rightarrow_R a'$ if $(a, a') \in \rightarrow_R$.
- $\twoheadrightarrow_R$ denotes the reflexive transitive closure of $\rightarrow_R$. (Multistep rewriting; 0 or more steps of $\rightarrow_R$)
- $=_R$ denotes the symmetric transitive closure of $\twoheadrightarrow_R$. (Smallest equivalence relation containing $\twoheadrightarrow_R$.)
  This is similar to β-reduction in λ-calculus, where we have $\rightarrow_\beta$, $\twoheadrightarrow_\beta$ and $=_\beta$.
- $a \in A$ is in $\rightarrow_R$-normal form if $\neg\exists b \in A(a \rightarrow_R b)$.

How can one prove the Church-Rosser property? (I)

DEFINITION The rewriting system $(A, \to_R)$ satisfies the Diamond Property, DP, if

$$\forall a, b_1, b_2 \in A(a \to_R b_1 \wedge a \to_R b_2 \implies \exists c \in A(b_1 \to_R c \wedge b_2 \to_R c)).$$

In a diagram:



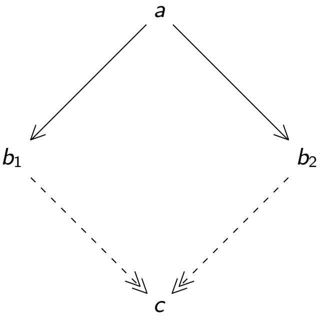
LEMMA DP($\to_R$) implies CR($\to_R$)

PROOF

How can one prove the Church-Rosser property? (II)

DEFINITION The rewriting system $(A, \to_R)$ satisfies the Weak Church-Rosser Property, WCR, if

$$\forall a, b_1, b_2 \in A(a \to_R b_1 \wedge a \to_R b_2 \implies \exists c \in A(b_1 \twoheadrightarrow_R c \wedge b_2 \twoheadrightarrow_R c)).$$



Note!: WCR($\to_R$) does not imply CR($\to_R$)

But we do have
NEWMAN'S LEMMA WCR($\to_R$) + SN($\to_R$) implies CR($\to_R$)

But for type theory, we need first CR($\to_\beta$), which will be used in the meta theory and in the proof of SN($\to_\beta$).

Intermezzo: proof of Newman's Lemma

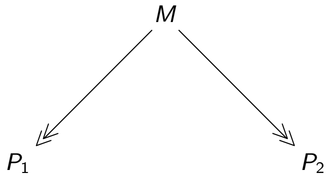
NEWMAN'S LEMMA WCR + SN implies CR

PROOF Constructive proof. By induction on $M \in \mathsf{SN}$, we prove that M is CR.

$$\frac{M \in \mathsf{NF}}{M \in \mathsf{SN}}\,(\text{base}) \qquad\qquad \frac{\forall P, (\text{if } M \to_R P \text{ then } P \in \mathsf{SN})}{M \in \mathsf{SN}}\,(\text{step})$$

Corollaries of the Church-Rosser property

THEOREM CR($\rightarrow_R$) implies UN($\rightarrow_R$) (Uniqueness of Normal forms)



If P_1 and P_2 are in normal form, then $P_1 = P_2$, due to CR.

THEOREM CR($\rightarrow_R$) + SN($\rightarrow_R$) implies $=_R$ is decidable.

PROOF: To decide $a =_R b$, just rewrite a and b until you find their normal forms a' and b'. Due to UN (which follows form CR), we have $a =_R b$ iff $a' = b'$.

NB. Decidability of $=_\beta$ is crucial for decidability of type checking! Remember the conversion rule:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}\; A =_\beta B$$

We prove CR(β) for untyped λ-calculus

Untyped λ-calculus

$$M, N ::= x \mid M\,N \mid \lambda x.M$$

Reduction (inductive definition):

$$\frac{}{(\lambda x.M)P \to_\beta M[x := P]}\ (\beta) \qquad \frac{M \to_\beta M'}{M\,P \to_\beta M'\,P}\ \text{(app-l)}$$

$$\frac{M \to_\beta M'}{\lambda x.M \to_\beta \lambda x.M'}\ (\lambda) \qquad \frac{M \to_\beta M'}{P\,M \to_\beta P\,M'}\ \text{(app-r)}$$

NB. DP(β) fails due to redex erasure or redex duplication:

$$(\lambda x.y)(\mathbf{I}\,\mathbf{I}) \qquad (\lambda x.y\,x\,x)(\mathbf{I}\,\mathbf{I})$$

Parallel reduction in untyped λ-calculus

We prove CR(β) using parallel reduction, a method due to Tait and Martin-Löf and refined by Takahashi.

Parallel reduction $M \Longrightarrow P$ allows to contract several redexes in M in one step. It can be defined inductively.

Definition

$$\frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{(\lambda x.M)P \Longrightarrow M'[x := P']}\ (\beta) \qquad \frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{M\,P \Longrightarrow M'\,P'}\ (\text{app})$$

$$\frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}\ (\lambda) \qquad \frac{}{x \Longrightarrow x}\ (\text{var})$$

Examples:

$$(\lambda x.y\,x\,x)(\mathbf{I}\,\mathbf{I}) \qquad (\lambda x.x\,(x\,\mathbf{I}))(\mathbf{I}\,\mathbf{I})$$

Properties of parallel reduction

$$\frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{(\lambda x.M)P \Longrightarrow M'[x := P']}\,(\beta) \qquad \frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{M\,P \Longrightarrow M'\,P'}\,(\mathsf{app})$$

$$\frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}\,(\lambda) \qquad \frac{}{x \Longrightarrow x}\,(\mathsf{var})$$

Theorem

1. $M \Longrightarrow M$
   The proof is by induction on M.
2. If $M \to_\beta P$, then $M \Longrightarrow P$
   The proof is by induction on the derivation, using (1).
3. If $M \Longrightarrow P$, then $M \twoheadrightarrow_\beta P$.
   The proof is by induction on the derivation.

Parallel reduction satisfies a strong Diamond Property (I)

Theorem

$$\forall M \, \exists Q \, \forall P \, (\text{if } M \Longrightarrow P \text{ then } P \Longrightarrow Q).$$

This immediately implies DP($\Longrightarrow$) (and thereby CR(β)).
We can even define this Q inductively from M; it will be called M^*.
So we have

$$\forall M, P \, (\text{if } M \Longrightarrow P \text{ then } P \Longrightarrow M^*).$$

Note: This implies $\forall M \, (M \Longrightarrow M^*)$.

Definition

$$\begin{array}{rcl} x^* & := & x \\ (\lambda x.M)^* & := & \lambda x.M^* \\ ((\lambda x.P)\, N)^* & := & P^*[x := N^*] \\ (M\, N)^* & := & M^*\, N^* \text{ if } M \neq \lambda x.P \text{ } (M \text{ is not a } \lambda\text{-abstraction}) \end{array}$$

Parallel reduction satisfies a strong Diamond Property (II)

THEOREM

$$\forall M, P\,(\text{if } M \Longrightarrow P \text{ then } P \Longrightarrow M^*).$$

PROOF by induction on the derivation of $M \Longrightarrow P$. There are 4 cases. We treat 3 of them.

case (1)

$$\frac{}{x \Longrightarrow x}\,(\text{var})$$

Then indeed $x \Longrightarrow x^*$ (because $x^* = x$).

case (2)

$$\frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}\,(\lambda)$$

IH: $M' \Longrightarrow M^*$. We need to prove: $\lambda x.M' \Longrightarrow (\lambda x.M)^*$
We have $(\lambda x.M)^* = \lambda x.M^*$.
$\lambda x.M' \Longrightarrow \lambda x.M^*$ follows immediately from IH and the definition of $\Longrightarrow$.

Parallel reduction satisfies a strong Diamond Property (IV)

THEOREM

$$\forall M, P\,(\text{if } M \Longrightarrow P \text{ then } P \Longrightarrow M^*).$$

PROOF continued

case (4)

$$\frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{(\lambda x.M)\,P \Longrightarrow M'[x := P']}$$

IH: $M' \Longrightarrow M^*$ and $P' \Longrightarrow P^*$.
We need to prove: $M'[x := P'] \Longrightarrow ((\lambda x.M)\,P)^* = M^*[x := P^*]$.

To prove this we need a separate

SUBSTITUTION LEMMA If $M \Longrightarrow M'$ and $P \Longrightarrow P'$, then $M[x := P] \Longrightarrow M'[x := P']$.

This is proved by induction on the structure of M.

DP($\Longrightarrow$) implies CR(β)

The proof that DP($\Longrightarrow$) implies CR(β) follows from the properties we have established:

1. If $M \rightarrow_\beta P$, then $M \Longrightarrow P$.
2. If $M \Longrightarrow P$, then $M \twoheadrightarrow_\beta P$.
3. If $M \Longrightarrow P$, then $P \Longrightarrow M^*$.

Yet another example

$$(\lambda z.z\, z)\,(\mathbf{I}\,(\mathbf{I}\,x))$$

The same example again

$$\begin{array}{rcl} x^* & := & x \\ (\lambda x.M)^* & := & \lambda x.M^* \\ (M\,N)^* & := & P^*[x := N^*] \text{ if } M = \lambda x.P \\ & := & M^*\,N^* \text{ otherwise.} \end{array}$$

$$(\lambda z.z\,z)\,(\mathbf{I}\,(\mathbf{I}\,x))$$

This is a flexible proof of Church-Rosser

- Methods works for proving CR for reduction in Combinatory Logic
- Methods works for proving CR for β on pseudo-terms of Pure Type Systems
- Method extends to typed lambda calculus with data types, for example natural numbers:

$$M, N := x \mid M\,N \mid \lambda x.M \mid 0 \mid \mathbf{suc}\,M \mid \mathbf{nrec}\,M\,N\,P$$

  with

$$\begin{array}{rcl} \mathbf{nrec}\,M\,N\,0 & \to & M \\ \mathbf{nrec}\,M\,N\,(\mathbf{suc}\,P) & \to & N\,P\,(\mathbf{nrec}\,M\,N\,P) \end{array}$$

- Method extends to η-reduction:

$$\lambda x.M\,x \to_\eta M \qquad \text{if } x \notin \mathrm{FV}(M)$$