Lambda-Calculus and Type Theory

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Obergurgl, Austria

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Lecture 11

Normalization by Evaluation

Previous Lecture

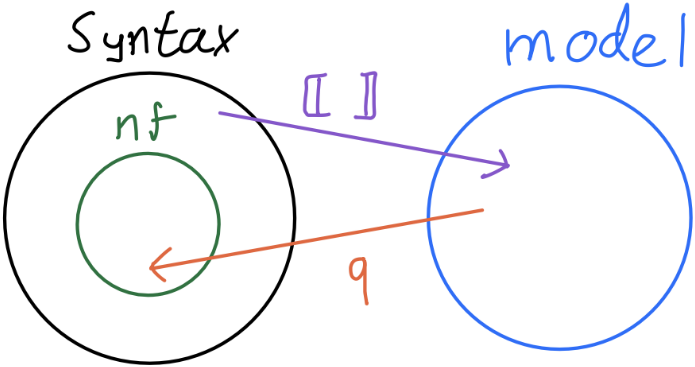
- We saw: the λ-calculus is strongly normalizing
- This gives us an algorithm to normalize λ-terms: find a β-redex and reduce it
- We shall look at another algorithm for finding normal forms

Reduction-Free Normalization

Topic of this lecture: **Normalization by Evaluation** (NbE)

- NbE does not work by finding redexes and reducing them
- Instead it works by **evaluating** terms in a suitable model and then **reifying them** back into the syntax
- The result is a normal form of the original term

Main Idea of NbE



This lecture

- We first look at normalizing monoid expressions.
- Then we illustrate how NbE works for the λ-calculus

Simple Example: Monoid Expressions

Let A be any set. The set $M(A)$ of monoid expressions over A is generated by the following grammar:

$$e := \quad u \quad | \quad v(a) \quad | \quad e_1 \cdot e_2$$

We also define an equivalence relation $\sim$ generated by:

$$u \cdot e \sim e$$

$$e \cdot u \sim e$$

$$(e_1 \cdot e_2) \cdot e_3 \sim e_1 \cdot (e_2 \cdot e_3)$$

Examples:

$$v(a), \quad v(a_1) \cdot (v(a_2) \cdot v(a_3)), \quad ((u \cdot v(a_1)) \cdot v(a_2)) \cdot (v(a_3) \cdot u)$$

Normal Forms of Monoid Expressions

Normal forms of monoid expressions are given by **lists.**
Lists l of A give rise to a monoid expression $\mathrm{incl}(l)$:

$$\mathrm{incl}([]) := u$$

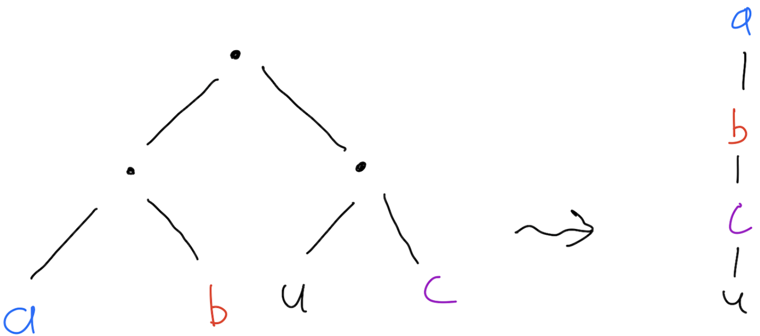
$$\mathrm{incl}(x :: xs) := v(x) \cdot \mathrm{incl}(xs)$$

The normal forms look as follows:

$$v(x_1) \cdot v(x_2) \cdot \ldots \cdot v(x_n) \cdot u$$

Note: everything in this lecture can be adapted so that our normal forms look like $v(x_1) \cdot v(x_2) \cdot \ldots \cdot v(x_n)$, but that is a bit more technically involved

Concretely



We normalize

$$(v(a) \cdot v(b)) \cdot (u \cdot (v(c)))$$

into

$$v(a) \cdot (v(b) \cdot (v(c) \cdot u))$$

Normalization function

We want to define a normalization function norm that sends expressions e to a list norm(e).
Correctness: $e_1 \sim e_2$ if and only if $\mathrm{norm}(e_1) = \mathrm{norm}(e_2)$
Deciding equality of monoid expressions: to check $e_1 \sim e_2$, it suffices to check whether $\mathrm{norm}(e_1) = \mathrm{norm}(e_2)$ as lists

What we will do

We define two normalization functions:

- Define a suitable model
- Define a reification function from the model to the syntax
- Their composition gives a normalization function

We will use two different models: lists and functions

Direct Proof of Normalization

Let e be an expression. We define $[\![e]\!]$:

$$[\![v(a)]\!] := a :: []$$

$$[\![u]\!] := []$$

$$[\![e_1 \cdot e_2]\!] := [\![e_1]\!] \mathbin{++} [\![e_2]\!]$$

The normalization function:

$$\mathrm{norm}(e) := \mathrm{incl}([\![e]\!])$$

Correctness

Theorem

We have: $e_1 \sim e_2$ *if and only if* $\text{norm}(e_1) = \text{norm}(e_2)$

This follows from the following lemmas:

Lemma

If $e_1 \sim e_2$*, then* $[\![e_1]\!] = [\![e_2]\!]$*.*

Lemma

For l_1 *and* l_2*, we have* $\text{incl}(l_1 +\!\!+\ l_2) = \text{incl}(l_1) \cdot \text{incl}(l_2)$*.*

Lemma

For all e*, we have* $e \sim \text{norm}(e)$*.*

An Example

Let's normalize: $((v(1) \cdot u) \cdot v(2)) \cdot v(3)$.

$$\mathrm{norm}(((v(1) \cdot u) \cdot v(2)) \cdot v(3))$$

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$$\begin{aligned} &\mathrm{norm}(((v(1) \cdot u) \cdot v(2)) \cdot v(3)) \\ &= [\![((v(1) \cdot u) \cdot v(2)) \cdot v(3)]\!](u) \end{aligned}$$

Unfold the definition

An Example

Let's normalize: $((v(1) \cdot u) \cdot v(2)) \cdot v(3)$.

$$\begin{aligned} &\mathrm{norm}(((v(1) \cdot u) \cdot v(2)) \cdot v(3)) \\ &= \mathrm{incl}([\![((v(1) \cdot u) \cdot v(2)) \cdot v(3)]\!]) \end{aligned}$$

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$$\begin{aligned} & \mathrm{norm}(((v(1) \cdot u) \cdot v(2)) \cdot v(3)) \\ &= \mathrm{incl}(\llbracket ((v(1) \cdot u) \cdot v(2)) \cdot v(3) \rrbracket) \\ &= \mathrm{incl}(\llbracket (v(1) \cdot u) \cdot v(2) \rrbracket \mathbin{+\!\!+} \llbracket v(3) \rrbracket) \end{aligned}$$

Use: $\llbracket e_1 \cdot e_2 \rrbracket := \llbracket e_1 \rrbracket \mathbin{+\!\!+} \llbracket e_2 \rrbracket$

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&= \mathrm{incl}(\llbracket v(1) \cdot u \rrbracket \mathbin{+\!\!+} \llbracket v(2) \rrbracket \mathbin{+\!\!+} \llbracket v(3) \rrbracket)
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Use: $[\![e_1 \cdot e_2]\!] := [\![e_1]\!] +\!\!+ [\![e_2]\!]$

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&= \mathrm{incl}((v(1) :: []) \mathbin{+\!\!+} [\![u]\!] \mathbin{+\!\!+} (v(2) :: []) \mathbin{+\!\!+} (v(3) :: []))
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Use $[\![v(a)]\!] := a :: []$

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Use $[\![u]\!] := []$

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&= v(1) \cdot (v(2) \cdot (v(3) \cdot u))
\end{aligned}
$$

Another Proof of Normalization

- The normalization function that we discussed, directly shows that we have a model given by normal forms
- However, often this is not feasible (for instance, for the λ-calculus)
- For this reason, we shall discuss another proof of normalization
- This time, the model is based on functions: every expression $e \in M(A)$ gives a function $M(A) \to M(A)$

Another Proof of Normalization: Interpretation

For $e \in M(A)$, we define a function $[\![e]\!] : M(A) \to M(A)$:

$$[\![v(a)]\!](e'') := v(a) \cdot e''$$

$$[\![u]\!](e'') := e''$$

$$[\![e \cdot e']\!](e'') := [\![e]\!]([\![e']\!](e''))$$

Another Proof of Normalization: Normalization

Given a function $f : M(A) \to M(A)$, define

$$\text{reify}(f) := f(u)$$

Now we define the normalization function as follows

$$\text{norm}(e) := \text{incl}(\llbracket e \rrbracket)$$

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Let's normalize: $((v(1) \cdot u) \cdot v(2)) \cdot v(3)$.

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Unfold the definition

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Use: $[\![e \cdot e']\!](e'') := [\![e]\!]([\![e']\!](e''))$

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Use: $[\![v(a)]\!](e'') := v(a) \cdot e''$

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&= [\![(v(1) \cdot u)]\!]([\![v(2)]\!](v(3) \cdot u)) \\
&= [\![(v(1) \cdot u)]\!](v(2) \cdot (v(3) \cdot u)) \\
&= [\![v(1)]\!]([\![u]\!](v(2) \cdot (v(3) \cdot u))) \\
&= [\![v(1)]\!](v(2) \cdot (v(3) \cdot u))
\end{aligned}$$

Use: $[\![u]\!](e'') := e''$

An Example

Let's normalize: $((v(1) \cdot u) \cdot v(2)) \cdot v(3)$.

$$
\begin{aligned}
&\mathrm{norm}(((v(1) \cdot u) \cdot v(2)) \cdot v(3)) \\
&= [\![((v(1) \cdot u) \cdot v(2)) \cdot v(3)]\!](u) \\
&= [\![(v(1) \cdot u) \cdot v(2)]\!]([\![v(3)]\!](u)) \\
&= [\![(v(1) \cdot u) \cdot v(2)]\!](v(3) \cdot u) \\
&= [\![(v(1) \cdot u)]\!]([\![v(2)]\!](v(3) \cdot u)) \\
&= [\![(v(1) \cdot u)]\!](v(2) \cdot (v(3) \cdot u)) \\
&= [\![v(1)]\!]([\![u]\!](v(2) \cdot (v(3) \cdot u))) \\
&= [\![v(1)]\!](v(2) \cdot (v(3) \cdot u))
\end{aligned}
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&= [\![v(1)]\!]([\![u]\!](v(2) \cdot (v(3) \cdot u))) \\
&= [\![v(1)]\!](v(2) \cdot (v(3) \cdot u)) \\
&= v(1) \cdot (v(2) \cdot (v(3) \cdot u))
\end{aligned}
$$

Use: $[\![v(a)]\!](e'') := v(a) \cdot e''$

An Example

Let's normalize: $((v(1) \cdot u) \cdot v(2)) \cdot v(3)$.

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&= [\![v(1)]\!](v(2) \cdot (v(3) \cdot u)) \\
&= v(1) \cdot (v(2) \cdot (v(3) \cdot u))
\end{aligned}
$$

Correctness

Theorem

We have: $e_1 \sim e_2$ *if and only if* $\text{norm}(e_1) = \text{norm}(e_2)$

This follows from the following lemmas:

Lemma

If $e_1 \sim e_2$*, then* $[\![e_1]\!] = [\![e_2]\!]$*.*

Lemma

For e_1 *and* e_2*, we have* $e_1 \cdot e_2 = [\![e_1]\!](e_2)$*.*

Lemma

For all e, we have $e \sim \text{norm}(e)$*.*

Recap

So, we did the following:

- We defined interpretations of monoid expressions: via lists and via functions
- We showed how to reify the interpretations back to expressions
- Result: a normalization function

This is **normalization by evaluation.**
It can be applied to the λ-calculus as well.

NbE for the λ-calculus

- We shall define a normalization function for simply-typed λ-terms
- The output is an η-long β-normal form

Overall structure is similar to how SN is proved, but there are differences:

- For SN, we proved a **predicate** on terms, while now we define a **function** on terms
- Throughout the proof, we shall use **contexts** to indicate the free variables of terms

η-expansion

The η-rule:

$$M \equiv \lambda x.M\, x$$

Two ways of using this:

- η-contraction: rewrite $\lambda x.M\, x$ to M
- η-expansion: rewrite M to $\lambda x.M\, x$

Neutral and Normal Forms: Idea

- To define normal forms of the STLC, we also need **neutral forms**
- Neutral form: we can apply it to a normal form to get another normal form
- Note: $\lambda x.x$ is not neutral

Neutral and Normal Forms: Definition

Neutral forms Ne_A:

- if x is a variable of type A, then $x \in \mathsf{Ne}_A$
- if $m \in \mathsf{Ne}_{A \to B}$ and $n \in \mathsf{Nf}_A$, then $m\,n \in \mathsf{Ne}_B$

Normal forms Nf_A:

- if $n \in \mathsf{Ne}_\sigma$, then $n \in \mathsf{Nf}_\sigma$ (here σ is a base type)
- if $n \in \mathsf{Nf}_B$, then $\lambda(x : A), n : \mathsf{Nf}_{A \to B}$

We write $\mathsf{Ne}_A(\Gamma)$ and $\mathsf{Nf}_A(\Gamma)$ for sets of neutral terms and normal terms whose variables are in Γ.

Examples of a Normal Form

Is the following term an η-long β-normal form?

$$\lambda(x : \sigma), x$$

Yes!

- Since x is a variable, $x \in \mathsf{Ne}_\sigma$
- Since σ is a base type, $x \in \mathsf{Nf}_\sigma$
- Hence, $\lambda(x : \sigma), x \in \mathsf{Nf}_{\sigma \to \sigma}$

Examples of a Normal Form

Is the following term an η-long β-normal form?

$$\lambda(f : A \to B), f$$

No!

- To check that $\lambda(f : A \to B), f \in \mathsf{Nf}_{(A\to B)\to(A\to B)}$, we need to check that $f \in \mathsf{Nf}_{A\to B}$
- Since f is a variable, we need to check that $f \in \mathsf{Nf}_{A\to B}$
- However, $f \notin \mathsf{Nf}_{A\to B}$, because f is not a λ-abstraction
- We can't use that f is a variable, because $\mathsf{Ne}_\sigma \subseteq \mathsf{Nf}_\sigma$ only for base types σ
- $A \to B$ is not a base type

Examples of a Normal Form

Is the following term an η-long β-normal form?

$$\lambda(f : A \to B)(x : A), f\ x$$

Yes!

- Goal: $\lambda(f : A \to B)(x : A), f\ x$ is in normal form
- Sufficient: $\lambda(x : A), f\ x$ is in normal form
- Sufficient: $f\ x$ is in normal form
- We need to check f is a **neutral form** and x is a normal form
- x is a normal form: it is a variable of a base type
- f is a neutral form, because it is a variable

Contexts

Definition

A **context** is given by a finite set of variable declarations such that each variable is declared at most once. Contexts are ordered by **inclusion**. The set of contexts is denoted by Con.

For instance, $\{x : A, y : B\}$.

Main Steps

To define NbE for the STLC, we take the following steps

- Define the model
- Define the interpretation
- Define reification

Model: Interpretation of Types

To prove strong normalization, we first define a predicate $[\![A]\!]$ for terms on types A

Concretely we defined:

- $[\![\sigma]\!]$: strongly normalizing terms of type σ
- $[\![A \to B]\!]$: the set $\{M \mid \forall_{N \in [\![A]\!]} MN \in [\![B]\!]\}$

For NbE, we do something similar, but

- we need to keep track of contexts
- we need to work proof relevant: instead of defining a predicate, we define a set

Model: Interpretation of Types

We interpret types A as a map $[\![A]\!] : \mathrm{Con} \to \mathrm{Set}$ together with functions $[\![A]\!]_{\Gamma_1,\Gamma_2} : [\![A]\!](\Gamma_1) \to [\![A]\!](\Gamma_2)$ whenever $\Gamma_1 \subseteq \Gamma_2$.

Definition

We interpret base types as follows: $[\![\sigma]\!](\Gamma) = \mathrm{Nf}_\sigma(\Gamma)$.
For function types: elements of $[\![A \to B]\!](\Gamma)$ consist of

- for all Γ' such that $\Gamma \subseteq \Gamma'$ a function $f^{\Gamma'} : [\![A]\!](\Gamma') \to [\![B]\!](\Gamma')$
- such that for all $\Gamma', \Gamma'' \in \mathrm{Con}$ with $\Gamma' \subseteq \Gamma''$ and all $x \in [\![A]\!](\Gamma')$, we have

$$f^{\Gamma''}([\![A]\!]_{\Gamma',\Gamma''}(x)) = [\![B]\!]_{\Gamma',\Gamma'}(f^{\Gamma'}(x))$$

Comparison

The case for the function type might seem mysterious, but compare the following:

SN: $\{M \mid \forall_{N \in [\![A]\!]} MN \in [\![B]\!]\}$

NbE: $f^{\Gamma'} : [\![A]\!](\Gamma') \to [\![B]\!](\Gamma')$

And we recall the previous lecture again

Proposition

If

- $x_1 : A_1, \ldots, x_n : A_n \vdash M : B$
- $N_1 \in [\![A_1]\!], \ldots N_n \in [\![A_n]\!]$,

Then $M[x_1 := N_1, \ldots x_n := N_n] \in [\![B]\!]$

For NbE: we need to give an **interpretation of terms**

The Model: Terms

Suppose, we have a term t of type A.
Given

- a context Γ containing the free variables of t,
- an element $\rho(x) : [\![B]\!](\Gamma)$ for each declaration $x : B$ in Γ,

we interpret t as an element $[\![t]\!]^{\Gamma}_{\rho}$ of $[\![A]\!](\Gamma)$.

The Model: Terms (Variables)

We define:

$$[\![x]\!]^{\Gamma}_{\rho} = \rho(x)$$

This works because $\rho(x) : [\![B]\!](\Gamma)$ where B is the type of x.

The Model: Terms (Application)

- Suppose $M : A \to B$ and $N : A$.
- Note $[\![M]\!]^{\Gamma}_{\rho}$ gives a function $f^{\Gamma'} : [\![A]\!](\Gamma') \to [\![B]\!](\Gamma')$ for all $\Gamma \subseteq \Gamma'$
- Also note: $[\![N]\!] : [\![A]\!](\Gamma)$

Define

$$[\![MN]\!]^{\Gamma}_{\rho} = [\![M]\!]^{\Gamma}_{\rho}([\![N]\!]^{\Gamma}_{\rho})$$

The Model: Terms (Abstraction)

- Suppose $M : B$
- Let Γ contain the free variables of $\lambda(x : A).M$
- ρ maps variable declarations $y : B$ in Γ to $\rho(y) : [\![B]\!](\Gamma)$

Goal: to define $[\![\lambda(x : A).M]\!]^{\Gamma}_{\rho}$, we need to give for all Γ' such that $\Gamma \subseteq \Gamma'$ a function $f^{\Gamma'} : [\![A]\!](\Gamma') \to [\![B]\!](\Gamma')$
So, suppose we have,

- a context Γ' such that $\Gamma \subseteq \Gamma'$
- $z : [\![A]\!](\Gamma')$

Define $\rho[x \mapsto z]$ to be ρ but sending x to z
Then

$$[\![M]\!]^{\Gamma}_{\rho[x \mapsto z]} : [\![B]\!](\Gamma)$$

So, we take

$$[\![B]\!]_{\Gamma,\Gamma'}([\![M]\!]^{\Gamma}_{\rho[x \mapsto z]}) : [\![B]\!](\Gamma')$$

Reification

Recall the following lemmas when proving strong normalization:

Lemma

For all strongly normalizing terms $N_1 : A_1, ..., N_k : A_k$ and variables $x : A_1 \to \ldots \to A_k \to B$, we have

$$xN_1 \ldots N_k : [\![B]\!]$$

Lemma

Every inhabitant of $[\![A]\!]$ is strongly normalizing

These were proven by mutual induction.

These are also needed for NbE.

Concretely: we define the **quote** and **unquote** functions

Reification

Lemma

For all contexts Γ *and types* A*, we have functions*

$$u_A^\Gamma : \mathrm{Ne}_A(\Gamma) \to [\![A]\!](\Gamma)$$

$$q_A^\Gamma : [\![A]\!](\Gamma) \to \mathrm{Nf}_A(\Gamma)$$

Proof.

Exercise! $\square$

u_A^Γ is called **unquote** and q_A^Γ is called **quote.**
Reification: given by q_A^Γ.

Normalization

Let $M : A$ be a term of type A free variables in Γ.
We define the normalization function

$$\mathrm{norm}(M) = q_A^\Gamma(\llbracket M \rrbracket_\rho^\Gamma)$$

where $\rho(x) = u_A^\Gamma(x)$.
Note that we need to use unquote here, because we need values in $\llbracket A \rrbracket(\Gamma)$ and not just variables.

Summary

Key points of this lecture:

- Normalization by evaluation is a different technique for normalizing terms
- It is not based on rewriting
- Instead it evaluates the term in a certain model and then reifies the result back to the syntax
- Possible for monoid expressions: one can use lists or functions
- Possible for the STLC: use sets indexed by contexts
- Many extensions of NbE are possible, for instance to dependent type theory
- NbE is also usable to implement proof assistants