Lambda-Calculus and Type Theory

ISR 2024

Obergurgl, Austria

Herman Geuvers & Niels van der Weide

Radboud University Nijmegen NL

Test

The maximum number of points per exercise is indicated in the margin. Maxium 100 points in total.
NB Typing derivations may be given in "flag style" or in "sequent style".
You can do exercises 7 and 8 either in Coq or on paper. In case you do these in Coq, use the `test.v` file on the website and send the completed file to `herman@cs.ru.nl` and `nweide@cs.ru.nl`

---

(10) 1. Give a typing derivation that shows that the following term is typable in simple type theory ($\lambda\!\to$) à la Church.

$$\lambda x : \alpha.\, \lambda y : ((\alpha \to \alpha) \to \alpha) \to \alpha.\, y\, (\lambda z : \alpha \to \alpha.\, z\, x)$$

(10) 2. Give a term M in simple type theory ($\lambda\!\to$) à la Church with type

$$(\gamma \to \beta) \to ((\alpha \to \beta) \to \alpha) \to \gamma \to \alpha.$$

Give a typing derivation that gives the type of M.

3. Consider the following term in the simply typed λ-calculus.

$$N \quad := \quad (\lambda y{:}\alpha \to \alpha.\, \lambda x{:}\alpha.\, y\,(y\,x))\,(\mathbf{I\,I})$$

(8) (a) Compute $m(N)$, where $m(-)$ is the measure of a term, as used in the proof of weak normalization (WN) for $\lambda\!\to$. Indicate which redex will be contracted according to the strategy described in the WN proof.

(3) (b) Now consider N as a term in the untyped λ-calculus. Calculate N^*. (You can just give N^*, you don't have to show a computation.)

(6) (c) Give all terms M for which we have $N \Rightarrow M$. (You don't have to give the derivations of $N \Rightarrow M$.)

(10) 4. Show that, in $\lambda 2$ à la Church,

$$\lambda x{:}\bot.\, x\,((\top \to \top \to \top) \to \top)\,(\lambda y{:}\top.\, y\,\top) \quad : \quad \bot \to \top,$$

where $\top := \forall\alpha : *.\alpha \to \alpha$ and $\bot := \forall\alpha : *.\alpha$. Give the typing derivation of your result.

Continue on the other side

(12) 5. In $\lambda 2$ à la Church, we have the well-known types of natural numbers and of lists over natural numbers:

$$\begin{aligned} \mathbb{N} &:= \forall \alpha{:}*.\alpha \to (\alpha \to \alpha) \to \alpha \\ \mathbb{L}_{\mathbb{N}} &:= \forall \alpha{:}*.\alpha \to (\mathbb{N} \to \alpha \to \alpha) \to \alpha. \end{aligned}$$

You may assume that the function $\mathsf{plus} : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ (that adds two natural numbers) is known. Define the function $\mathbb{L}\mathsf{plus} : \mathbb{L}_{\mathbb{N}} \to \mathbb{N}$ that adds the numbers in a list.

(10) 6. In the system λP, give a term of type $\Pi x{:}A.R\,(f\,a)\,(f(f\,x))$ in the context

$$\begin{aligned} \Gamma := \; & A : *, a : A, R : A \to A \to *, Q : A \to A \to *, f : A \to A, \\ & p : \Pi x{:}A.Q\,a\,(f\,x), \quad q : \Pi x, y{:}A.Q\,x\,y \to R\,x\,y, \\ & r : \Pi x, y{:}A.R\,x\,y \to R\,(f\,x)\,(f\,y). \end{aligned}$$

That is, solve the following type inhabitation problem in λP:

$$\Gamma \vdash ? : \Pi x{:}A.R\,(f\,a)\,(f(f\,x))$$

Also give a typing derivation in short version (where you don't have to establish the well-formedness of the types themselves).

(15) 7. [You can also do this exercise in Coq] In the system λP, give a term of type

$$\Pi x, y{:}A.R\,x\,y \to R\,(f\,x)\,x$$

in the context

$$\begin{aligned} \Gamma := \; & A : *, R : A \to A \to *, f : A \to A \\ & h : \Pi x, y{:}A.R\,x\,y \to R\,(f\,y)\,x \\ & t : \Pi x, y, z{:}A.R\,x\,y \to R\,(f\,y)\,z \to R\,x\,z \end{aligned}$$

You don't have to give a typing derivation!

8. [You can also do this exercise in Coq] In Lecture 7 we discussed Higher order logic and the Calculus of Constructions. We defined, given $A : *, P : A \to *, f : A \to A$, the *smallest subset of A containing P and closed under f* as follows.

$$S := \lambda y : A.\forall Q : A \to *.(P \subseteq Q) \to (\forall x : A.Q\,x \to Q\,(f\,x)) \to Q\,y$$

where $P \subseteq Q := \forall x : A.P\,x \to Q\,x$.
In the lecture we proved that S is closed under f.

(8) (a) Prove that S contains P (that is, give a term of type $P \subseteq S$).

(8) (b) Prove that S is indeed the smallest such (that is give a term that proves that for all Q that contain P and are closed under f, we have $S \subseteq Q$).

You don't have to give a typing derivation!

END