Exercises Coalgebra for Lecture 10

The exercises labeled with **(*)** are optional and more advanced.

1. Prove the following equalities over languages $L, K, M \in 2^{A^*}$ using bisimulations (up to congruence):
   (a) $L \cdot 1 = L = 1 \cdot L$
   (b) $LL^* + 1 = L^*$
   (c) Let $A = \{a, b\}$; prove that $(a + 1)^* = a^*$.
   (d) $L(K + M) = LK + LM$
   (e) $(L^*)^* = L^*$
   (f) $L^*L^* = L^*$

2. **(*)** Re-prove the last two identities in the previous exercise using the axioms of Kleene algebra, and while you're at it, prove that $(L^*K)^*L^* = (L + K)^*$ either in KA or with bisimulations (or both).

3. In the lecture, we have characterized the output and derivative of $L + K$, LK and L^* in terms of the output and derivatives of L and K.
   (a) Give a similar characterization for intersection $L \cap K$ and complement $\overline{L}$, with the latter defined by $\overline{L} = \{w \mid w \notin L\}$.
   (b) The *shuffle* operation is defined on words w, v inductively as follows: $w \circledcirc \varepsilon = \varepsilon \circledcirc w = w$ and $aw \circledcirc bv = a(w \circledcirc bv) + b(aw \circledcirc v)$ for any alphabet letters a, b. This is extended to languages L, K as $L \circledcirc K = \sum_{w \in L, v \in K} w \circledcirc v$. Give a characterization of output and derivative for the shuffle. Prove that your answer is correct.
   (c) Use your result in the previous exercise to prove that $L \circledcirc K = K \circledcirc L$ for all $L, K \in 2^{A^*}$.
   (d) We define the set of *extended regular expressions* EExp by

   $$r \quad ::= \quad r + r \mid r \cdot r \mid r^* \mid \overline{r} \mid r \sqcap r \mid a \mid 1 \mid 0$$

   The semantics $L \colon \mathsf{EExp} \to 2^{A^*}$ is defined by extending the semantics of regular expressions with the cases of $\overline{r}$ and $r \sqcap s$ using complement and intersection. Define asuitable deterministic automaton over EExp, such that the language semantics coincides with L.
   (e) **(*)** There are unique languages L and K such that

   $$L = \{a\}L\{a\} + \{b\}L\{b\} + \{a\} + \{b\} + 1$$
   $$K = \{a\}K\{a\} + \{b\}K\{b\} + \{a\}A^*\{b\} + \{b\}A^*\{a\}$$

   L is the language of *palindromes*: words which are equal to their own reverse. The aim of this exercise is to prove that K is the language of all *non*-palindromes, by showing that the relation $R = \{(\overline{L}, K)\}$ is a bisimulation up to congruence. Hint: to relate the derivatives, use that $\overline{L\{a\}} = (\overline{L}\{a\} + A^*\{b\} + 1)$ (which holds for every language L).

4. **(*)** Prove that for all languages $L, K \in 2^{A^*}$ and all $a \in A$:

$$\begin{aligned} (L + K)_a &= L_a + K_a \\ (LK)_a &= L_a K + L(\varepsilon) K_a \\ (L^*)_a &= L_a L^* \end{aligned}$$