Coalgebra lecture 12, exercise 5

December 3, 2018

Let $(X, \to)$ be a labelled transition system over A. Let $A^\infty = A^\omega \cup A^*$ be the set of streams and (finite) words over A. The empty word is denoted by ε. Consider the following rules, involving a relation $\downarrow \subseteq X \times A^\infty$.

$$\frac{}{x \downarrow \varepsilon} \qquad \frac{x \xrightarrow{a} y \quad y \downarrow w}{x \downarrow aw} \tag{1}$$

(for all $a \in A$, $w \in A^\infty$).

Let $\mathsf{Rel}_{X,A^\infty}$ be the set of relations of the form $R \subseteq X \times A^\infty$, partially ordered by subset inclusion $\subseteq$. This partial order is a complete lattice.

Given $x \in X$ and $w \in A^\infty$, we say w is a *trace* of x if there is a path from x labelled by w, that is, a path $x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \ldots$ such that $x_1 = x$ and $w = a_1a_2a_3\ldots$. If $w \in A^*$ then we call this a *finite* trace, if $w \in A^\omega$ we call it an *infinite* trace.

(a) Describe the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ of an arbitrary set $S \subseteq \mathsf{Rel}_{X,A^\infty}$, and give the top and bottom elements of the lattice (you don't have to give a proof).

Solution. The least upper bound of a set $S \subseteq \mathsf{Rel}_{X,A^\infty}$ is given by union: $\bigvee S = \bigcup_{R \in S} R$, the greatest lower bound by intersection $\bigwedge S = \bigcap_{R \in S} R$, the top element by $X \times A^\infty$ and the bottom element by $\emptyset$.

(b) Formulate the rules (1) in terms of a function $b \colon \mathsf{Rel}_{X,A^\infty} \to \mathsf{Rel}_{X,A^\infty}$. Show that your function is monotone.

Solution.

$$b(R) = \{(x, w) \mid w = \varepsilon \text{ or } \exists a \in A, v \in A^\infty, y \in X.\, w = av, x \xrightarrow{a} y \text{ and } (v, y) \in R\}\,.$$

For monotonicity, suppose $R \subseteq S$; we should prove that $b(R) \subseteq b(S)$. Let $(x, w) \in b(R)$. If $w = \varepsilon$ then $(x, w) \in b(S)$ is immediate from the definition; if $w = av$ then $x \xrightarrow{a} y$ for some y with $(y, v) \in R$. But $R \subseteq S$, so $(y, v) \in S$, and it follows that $(x, w) \in b(S)$.

(c) What is a pre-fixed point of b? And what is the least fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

Solution. A relation R is a pre-fixed point of b if $b(R) \subseteq R$. Concretely, this means

- $(x, \varepsilon) \in R$ for all $x \in X$, and
- if $x \xrightarrow{a} y$ and $(w, y) \in R$ then $(x, aw) \in R$.

The least fixed point is also the least pre-fixed point: the least relation satisfying the above conditions. It is given by the relation $\{(x, w) \mid w \text{ is a finite trace of } x\}$.

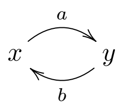
(d) What is a post-fixed point of b? And what is the greatest fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

Solution. A relation R is a post-fixed point of b if $R \subseteq b(R)$. This means: $\forall (x, w) \in R$,

- $w = \varepsilon$, or
- $w = av$, $x \xrightarrow{a} y$ and $(y, v) \in R$.

The greatest fixed point is the greatest post-fixed point, and is given by the relation $\{(x, w) \mid w \text{ is a (finite or infinite) trace of } x\}$.

(e) Use your answer to one of the previous questions to show that, in the transition system below, every finite trace of x is a prefix of the stream $ababab\ldots$.



Solution. As explained above, the finite traces are given by $\mathsf{lfp}(b)$, the least fixed point of b. The induction proof principle (following from Knaster-Tarski) states that

$$\frac{b(R) \subseteq R}{\mathsf{lfp}(b) \subseteq R} \tag{2}$$

for every $R \in \mathsf{Rel}_{X,A^\omega}$. We choose

$$R = \{(x, w) \mid w \text{ is a prefix of } (ab)^\omega\} \cup \{(y, w) \mid w \text{ is a prefix of } (ba)^\omega\}\,.$$

It is easy to check that R is indeed a pre-fixed point of b, hence $\mathsf{lfp}(b) \subseteq R$; so whenever w is a trace of x, we have $(x, w) \in R$, which means w is a prefix of $(ab)^\omega$.

(f) Use your answer to one of the previous questions to show that, in the transition system above, the stream $ababab\ldots$ is an infinite trace of x.

Solution. The relation of finite and infinite traces is given by $\mathsf{gfp}(b)$, the greatest fixed point of b. Thus, it suffices to show that $(x, (ab)^\omega) \in \mathsf{gfp}(b)$. To do so, we use the coinductive proof priniciple:

$$\frac{R \subseteq b(R)}{R \subseteq \mathsf{gfp}(b)} \tag{3}$$

and choose

$$R = \{(x, (ab)^\omega), (y, (ba)^\omega)\}\,.$$

It is easy to check that R is a post-fixed point of b, so that $R \subseteq \mathsf{gfp}(b)$, which in particular means $(x, (ab)^\omega) \in \mathsf{gfp}(b)$.

(g) Suppose that our transition system is finitely branching, meaning that for each $x \in X$, the set $\{y \mid x \xrightarrow{a} y \text{ for some } a\}$ is finite. Consider the relation $\| \subseteq X \times A^\infty$, given by: $x \| w$ iff there are infinitely many prefixes that are finite traces of x. Prove that for all $x \in X$ and $w \in A^\infty$: if $x \| w$, then w is an infinite trace of x.

Solution. We are going to prove that $\|$ is a post-fixed point of b. Again by coinduction (3), we then obtain that $x \| w$ implies $(x, w) \in \mathsf{gfp}(b)$, meaning that w is a trace of x.

Suppose $x \| w$. We need to prove that $(x, w) \in b(\|)$. First observe that $w \in A^\omega$, since it has infinitely many prefixes. Hence $w = av$ for some $a \in A$. Now, by definition of traces, for every trace of x of the form au there exists a state y such that $x \xrightarrow{a} y$, and u is a trace of y. In particular, there are infinitely many prefixes of $w = av$ which are traces of x, so there are infinitely many prefixes of v which are traces of y for some state y with $x \xrightarrow{a} y$. But there are only *finitely* many y such that $x \xrightarrow{a} y$, by the assumption that the LTS is finitely branching. Hence, there must be a state y such that $x \xrightarrow{a} y$ and infinitely many prefixes of v are a trace of y. Thus $y \| v$, and $(x, y) \in b(\|)$.